Physics 786, Spring 2014  
Problem Set 3 Due Wednesday, February 19, 2014.

1. Gravitational Waves

a) Suppose that a gravitational plane wave has wavevector \( k^\mu = (k, 0, 0, k) \) and polarization tensor \( \epsilon_{\mu\nu} \) satisfying the harmonic gauge condition,

\[
k^\mu \epsilon_{\mu\nu} = \frac{1}{2} k_\nu \epsilon_{\mu}^\nu.
\]

Show that the components of \( \epsilon_{\mu\nu} \) are related as follows:

\[
\begin{align*}
\epsilon_{01} &= -\epsilon_{31} , \\
\epsilon_{02} &= -\epsilon_{32} , \\
\epsilon_{22} &= -\epsilon_{11} , \\
\epsilon_{03} &= -\frac{1}{2} (\epsilon_{33} + \epsilon_{00}).
\end{align*}
\]

b) Show that by making a gauge transformation which preserves the harmonic gauge condition, \( \epsilon_{\mu\nu} \) from part (a) can be put in the form,

\[
\epsilon_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \epsilon_+ & \epsilon_\times & 0 \\
0 & \epsilon_\times & -\epsilon_+ & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

c) Consider a wavepacket \( h_{\mu\nu} = f(z-t)\epsilon_{\mu\nu} + \text{c.c.} \), with \( f(z-t) \) a positive wavefunction spread over some width \( L \), and \( \epsilon_{\mu\nu} \) as in part (b). Three particles in the \( x^1 - x^2 \) plane have spatial coordinates \( \mathbf{x}_1 = (0, 0, 0) \), \( \mathbf{x}_2 = (a, a, 0) \), and \( \mathbf{x}_3 = (-a, a, 0) \). We may assume \( a \ll L \).

Calculate the time evolution of the proper distance between pairs of particles 1 and 2, and between particles 1 and 3, as the gravitational wave passes. Compare the time evolution of these two proper distances.

d) Show that under a rotation of the coordinates by angle \( \theta \) about the \( x^3 \)-axis, the combination \( \epsilon_+ \pm i\epsilon_\times \) transforms as,

\[
\epsilon_+ \pm i\epsilon_\times \rightarrow e^{\pm 2i\theta} (\epsilon_+ \pm i\epsilon_\times).
\]

Hence, gravitational waves have helicity \( \pm 2 \).