Covariant Derivative

Vector fields on a manifold live in the tangent space, i.e. in the tangent plane at each pt. in the manifold.

\[ \tilde{\nabla}(x) = W^m(x) \tilde{e}_m(x) \]

Tangent plane at \( x \)

This discussion is the same in space or spacetime so we use Greek indices for generality.

Ordinary derivatives of \( \tilde{W}(x) \) do not generally remain in the tangent space:

\[ \partial \tilde{W}(x) = \partial_v \left( W^m \tilde{e}_m \right) = (\partial_v W^m) \tilde{e}_m + W^m \partial_v \tilde{e}_m, \]

or, using \( \partial \tilde{e}_m = \Gamma^m_{\nu \mu} \tilde{e}_\nu + K_{\nu \mu} \tilde{n} \),

\[ \partial \tilde{W} = (\partial_v W^m) \tilde{e}_m + W^m \Gamma^m_{\nu \mu} \tilde{e}_\nu + W^m K_{\nu \mu} \tilde{n} \]

\[ = (\partial_v W^m + \Gamma^m_{\nu \mu} W^\nu) \tilde{e}_m + W^m K_{\nu \mu} \tilde{n} \]

The projection of \( \partial \tilde{W} \) onto the tangent plane defines the covariant derivative of \( \tilde{W}(x) \):

\[ D_v \tilde{W} = (\partial_v W^m + \Gamma^m_{\nu \mu} W^\nu) \tilde{e}_m = (D_v W^m) \tilde{e}_m \]

Note that \( D_v W^m \) depends in general on all components of \( W^m \), not just the component \( m = 1 \).

\[ D_v \tilde{W} \]

The covariant derivative lives in the tangent space, and \( D_v W^m \) transforms as a tensor. This is another way to introduce the covariant derivative: we ask for a derivative that transforms covariantly under coordinate transformations.
The Affine connection is not a tensor under
general coordinate transformations.

\[ \Gamma^L_{MN} \equiv \frac{\partial X^L}{\partial x^M} \frac{\partial X^M}{\partial x^N} \quad \text{is a locally metric coordinate system.} \]

In the coordinate system \( x' \),

\[ \Gamma'^L_{MN} = \frac{\partial x'^L}{\partial x^M} \frac{\partial x'^M}{\partial x^N} \]

chain rule:

\[ \frac{\partial}{\partial x^N} \frac{\partial}{\partial x^M} \left( \frac{\partial x^L}{\partial x^N} \frac{\partial x^N}{\partial x^M} \right) \]

\[ = \frac{\partial x'^L}{\partial x^M} \frac{\partial x'^M}{\partial x^N} \left[ \frac{\partial x^L}{\partial x^N} \frac{\partial x^N}{\partial x^M} + \frac{\partial x^L}{\partial x^N} \frac{\partial x^N}{\partial x^M} \right] \]

\[ \frac{\partial^2}{\partial x^N \partial x^M} \left( \right) = \frac{\partial x'^L}{\partial x^M} \frac{\partial x'^M}{\partial x^N} \left[ \frac{\partial x^L}{\partial x^N} \frac{\partial x^N}{\partial x^M} + \frac{\partial x^L}{\partial x^N} \frac{\partial x^N}{\partial x^M} \right] \]

\[ \frac{\partial}{\partial x^N} \frac{\partial}{\partial x^M} \left( \right) \]

Differentiation of a tensor does not generally yield
another tensor.

Under the transformations \( x \rightarrow x' \), \( V^m = \frac{\partial x'^m}{\partial x^j} V^j \) for vector \( V^m \).

\[ \frac{\partial V'^m}{\partial x^j} = \frac{\partial x'^m}{\partial x^j} \frac{\partial x^j}{\partial x^i} \frac{\partial V^i}{\partial x^m} + \frac{\partial^2 x'^m}{\partial x^j \partial x^i} \frac{\partial V^i}{\partial x^m} \]

\[ \text{Tensor-like transforms, non-tensorial.} \]
However, the combination \( D^j V^m = V^m_{;j} = \frac{\partial V^m}{\partial x^j} + \Gamma_{jk}^m V^k \), the covariant derivative, is a tensor:

\[ V^m_{;j} = \frac{\partial x^m}{\partial x^j} \frac{\partial x^p}{\partial x^j} V^p \]

To show this we rewrite the transformates of \( \Gamma_{mn}^j \) in a different way.

Use \( \frac{\partial x^m}{\partial x^j} \frac{\partial x^p}{\partial x^j} = \delta^p_m \)

\[ \frac{\partial}{\partial x^j} \frac{\partial x^m}{\partial x^j} = 0 \]

\[ \rightarrow \Gamma_{mn}^j = \frac{\partial x^m}{\partial x^j} \frac{\partial x^p}{\partial x^k} \frac{\partial x^j}{\partial x^k} - \frac{\partial x^m}{\partial x^j} \frac{\partial x^p}{\partial x^k} \frac{\partial x^p}{\partial x^j} \frac{\partial x^k}{\partial x^j} \frac{\partial x^r}{\partial x^k} \frac{\partial x^s}{\partial x^r} V_{ik}^j \]

It is now straightforward to show that the non-tensor like term in the transformates of \( \Gamma_{mn}^j \) cancels the non-tensor like term in the transformates of \( \partial^j V^m \) in the combination \( D^j V^m + \Gamma_{jk}^m V^k \). (Exercise)

Similarly, the covariant derivative of a covariant vector

\[ D^j V_m = V_{m;j} = \frac{\partial V_m}{\partial x^j} - \Gamma_{jk}^m V^j \]

Under a coordinate transformation, \( D^j V_m \) transforms as a tensor:

\[ V'_{m;j} = \frac{\partial x^0}{\partial x'^m} \frac{\partial x^0}{\partial x'^j} V_{i;0} \] (Exercise)
In general, covariant derivatives of tensors involve a sum of terms, each involving one factor of $\Gamma^a_{\mu \nu}$, one for each index on the tensor.

**Example:** $T^a_{\mu \sigma} \lambda^\rho = \frac{\partial}{\partial x^\lambda} T^a_{\mu \sigma} \lambda + \Gamma^a_{\mu \nu} T^\nu_{\sigma \lambda} + \Gamma^a_{\mu \sigma} T^\nu_{\nu \lambda} - \Gamma^a_{\lambda \rho} T^\mu_{\nu \sigma}$

**Exercise:** Check that $T^a_{\mu \sigma} \lambda^\rho$ is a tensor.

**Properties of Covariant Derivatives**

1) $(\alpha A^\mu + \beta B^\mu)_j^\rho = \alpha A^\mu_j^\rho + \beta B^\mu_j^\rho$ \hspace{1cm} (linearity)

2) $(A^\mu B^\nu)_j^\rho = A^\mu_j^\rho B^\nu + A^\nu_j^\rho B^\mu$ \hspace{1cm} (Leibniz rule)

3) $T^a_{\mu \lambda} \lambda^\rho = \frac{\partial}{\partial x^\lambda} T^a_{\mu \lambda} \lambda + \Gamma^a_{\mu \nu} T^\nu_{\lambda \lambda} \lambda^\rho$ \hspace{1cm} (Denominator of $\Gamma^a_{\mu \nu}$)

**Covariant Differentiability of the Metric**

$\nabla^\mu \nabla_{\tau} = \frac{\partial}{\partial x^\lambda} - \Gamma^\rho_{\tau \mu} g_{\rho \sigma} + \nabla_{\nu} g_{\nu \rho}$

$= 0$ (using definition of $\nabla^\mu$ in terms of $\Gamma^\rho_{\tau \mu}$)

We can also show this by considering a locally inertial coordinate system, in which $\frac{\partial}{\partial x^\lambda} A^\nu = 0$ at some point. But $\nabla^\mu \nabla_{\tau}$ is a tensor, so in any general coordinate system, $\nabla^\mu \nabla_{\tau}$ remains zero. 
Similarly, \( g^{\mu \nu} \delta_{ij} = 0 \)
\[
\delta_{ij} = 0
\]

* Covariant Differentiation Commutes with Raising Lowering Indices
\[
(g^{\mu \nu} V_{j})_{i} = g^{\mu \rho} g_{\nu \sigma} V_{\rho} + g^{\mu \nu} V_{i j}
\]
\[
= g^{\mu \nu} V_{i j}
\]

Special Cases of Covariant Differentiation

Covariant Derivative of a Scalar \( S \):
\[
S_{\nu} = \frac{\partial S}{\partial x^{\nu}}
\]

Covariant Curl: Recall \( V_{\mu ; \nu} = \frac{\partial V_{\mu}}{\partial x^{\nu}} - \Gamma_{\mu \nu}^{\rho} V_{\rho} \)
\( \Delta \) symmetric in \( \mu \nu \).

Curl:
\[
V_{\mu ; \nu} - V_{\nu ; \mu} = \frac{\partial V_{\mu}}{\partial x^{\nu}} - \frac{\partial V_{\nu}}{\partial x^{\mu}} = \text{ordinary curl.}
\]

The covariant divergence of a covariant vector can be written in terms of \( g = \det(g_{\mu \nu}) \) using the following identity:
\[
\text{Tr} \left\{ M^{-1}(x) \frac{\partial}{\partial x^{\nu}} M(x) \right\} = \frac{\partial}{\partial x^{\nu}} \ln \det M(x)
\]

Proof: If \( x^{\lambda} \rightarrow x^{\lambda} + \delta x^{\lambda} \), then
\[
\delta \ln \det M = \ln \det (M + \delta M) - \ln \det M
\]
\[ \delta \ln \det M = \ln \left( \frac{\det (M + \delta M)}{\det M} \right) \]
\[ = \ln \det (M^{-1}(M + \delta M)) \]
\[ = \ln \det (1 + M^{-1} \delta M) \]
\[ \approx \ln (1 + \text{Tr} M^{-1} \delta M) \]
\[ \approx \text{Tr} M^{-1} \delta M \]

\[ \lim_{\delta x \to 0} \frac{\delta \ln \det M}{\delta x} = \frac{\partial}{\partial x^\alpha} \ln \det M \]
\[ = \text{Tr} \left( M^{-1} \frac{\partial M}{\partial x^\alpha} \right) \]

With \( M = g_{\mu \nu} \), \( g_{\mu \nu} \frac{\partial}{\partial x^\alpha} g_{\mu \nu} = \frac{\partial}{\partial x^\alpha} \ln g \)
\[ = \frac{2}{g} \frac{\partial}{\partial x^\alpha} \sqrt{g} \]

Then \( R^\alpha_{\mu \nu} = \frac{1}{2} g^{\alpha \sigma} \left( \partial_\mu g_{\sigma \nu} + \partial_\nu g_{\sigma \mu} - \partial_\sigma g_{\mu \nu} \right) \]
\[ = \frac{1}{2} g^{\alpha \sigma} \partial_\nu g_{\mu \sigma} \]

\[ R^\alpha_{\mu \nu} = \frac{1}{\sqrt{g}} \partial_\nu \sqrt{g} \]
\[ \Rightarrow V^\alpha_{\mu \nu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \sqrt{g} V^\alpha \]
Covariant Laplacian/ D'Alembertian

If \( \phi(x) \) is a scalar, \( \Phi_{\mu}^{\nu} = (g^{\mu\nu} \partial_{\nu} \phi)_{,\mu} \)

\[
\Phi_{\mu}^{\nu} = \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \partial_{\nu} \phi)
\]

In flat space these formulas allow us to compute the gradient, divergence, and curl in arbitrary coordinates.

Example: Laplacian in spherical coordinates

\[
d s^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2
\]

\[
g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}
\]

\[
g = \det g_{\mu\nu} = r^4 \sin^2 \theta
\]

\[
\nabla^2 \phi = \frac{1}{\sqrt{g}} \partial_{\mu} \left( \sqrt{g} g^{\mu\nu} \partial_{\nu} \phi \right)
\]

\[
= \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( r^2 \sin \theta \cdot \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} \right) \right. \\
\left. + \frac{\partial}{\partial \phi} \left( r^2 \sin \theta \cdot \frac{1}{r^2 \sin \theta} \frac{\partial \phi}{\partial \phi} \right) \right\}
\]

\[
= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right)
\]

\[
+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2}
\]
Example: Divergence in 2D Polar coordinates.

\( \partial_m V^m \) is not a scalar under general coordinate transformations. \( \partial_\alpha V^\alpha \) is a scalar.

Polar coordinates:

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  \cos \theta &= \frac{x}{\sqrt{x^2 + y^2}}
\end{align*}
\]

\[
V^m = \frac{\partial V^m}{\partial x^\nu} V^\nu.
\]

Let \((x, y)\) be the unprimed coordinates, \((r, \theta)\) the primed coordinates.

\[
V^r = \frac{\partial}{\partial x} V^x + \frac{\partial}{\partial y} V^y = \cos \theta V^x + \sin \theta V^y.
\]

\[
V^\theta = \frac{\partial}{\partial x} V^x + \frac{\partial}{\partial y} V^y = -\frac{\sin \theta}{r} V^x + \frac{\cos \theta}{r} V^y.
\]

\[
\partial_m V^m = \frac{1}{r} \partial_r (r V^r) + \frac{1}{r} \partial_\theta V^\theta.
\]

\[
\begin{align*}
d s^2 &= dr^2 + r^2 d\theta^2 \quad \Rightarrow \quad g = r^2 \quad (dx^\alpha dx^\beta = g_{\alpha \beta} x^\alpha x^\beta)
\end{align*}
\]

\[
\partial_m V^m = \frac{1}{r} \left[ \frac{\partial}{\partial r} (r V^r) + r \frac{\partial}{\partial \theta} V^\theta \right]
\]

\[
= \frac{1}{r} \frac{\partial}{\partial r} (r V^r) + \frac{\partial}{\partial \theta} V^\theta
\]

\( \partial_m V^m \) in polar coordinates.

Sometimes \( V^\theta \) is rescaled by \( r \) so that the dimension of \( V^\theta \) and \( V^r \) are the same. If \( V^\theta = r V^\theta \), then

\[
\partial_m V^m = \frac{1}{r} \frac{\partial}{\partial r} (r V^r) + \frac{1}{r} \frac{\partial}{\partial \theta} V^\theta.
\]