Plane wave solutions

Assume $T_{\mu\nu} = 0$.

\[ \left\{ \begin{array}{l}
\Box \Box h_{\mu\nu} = 0 \\
\Box \Box h_{\mu\nu} = \Box h_{\mu\nu} = h_{\mu\nu} = 0
\end{array} \right. \]

Linearized Einstein equations in harmonic gauge

We look for plane wave solutions to the set of equations (1).

Plane wave solutions have the form

\[ h_{\mu\nu}(x) = \varepsilon_{\mu\nu} \exp(ik \cdot x) + \varepsilon_{\mu\nu}^{*} \exp(-ik \cdot x) \]

where $k \cdot x = k_{\mu} x^{\mu} = k^{\nu} x_{\nu}$, $\varepsilon_{\mu\nu}$ = polarized field tensor, can be complex.

Consider derivatives of $\exp(ik \cdot x)$:

\[ \partial_{\alpha} \exp(ik \cdot x) = \frac{\partial}{\partial x^{\alpha}} \exp(ik \cdot x) \]

\[ = ik_{\alpha} \frac{\partial}{\partial x^{\alpha}} \exp(ik \cdot x) \]

\[ = ik_{\alpha} \delta^{\alpha}_{\mu} \exp(ik \cdot x) \]

\[ = ik_{\alpha} \exp(ik \cdot x) \]

Similarly, \( \Box \Box \exp(ik \cdot x) = -k_{\alpha} k^{\alpha} \exp(ik \cdot x) \).

Then the equations (1) imply

\[ \left\{ \begin{array}{l}
-k_{\alpha} k^{\alpha} h_{\mu\nu} = 0 \implies \frac{\partial}{\partial x^{\alpha}} h_{\mu\nu} = 0 \\
k_{\mu} \varepsilon_{\mu\nu}^{\*} = \frac{1}{2} k_{\nu} \varepsilon_{\mu\nu}
\end{array} \right. \]

Consider the gauge transformations:

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + 2 \varepsilon_{\mu} \xi^{\nu} + 2 \xi_{\nu} \varepsilon_{\mu} \]

\[ g^{\mu}(x) = -i \varepsilon_{\mu} \exp(ik \cdot x) + i \varepsilon_{\mu}^{*} \exp(-ik \cdot x) \]

This is equivalent to \( \varepsilon_{\mu\nu} \rightarrow \varepsilon_{\mu\nu} + k_{\mu} \varepsilon_{\nu} + k_{\nu} \varepsilon_{\mu} \).
Under this class of gauge transformations,
\[ k^m E^m \rightarrow k^m E^m + k^m k^n E^n (k^m E^m) k^n \]
\[ \frac{1}{2} k^m E^m \rightarrow \frac{1}{2} k^m E^m + \frac{1}{2} k^m k^n (k^m E^m) \]
\[ = \frac{1}{2} k^m E + k^m(k^m E) \]

\[ 0 = k^m E^m - \frac{1}{2} k^m E^m \rightarrow k^m E^m - \frac{1}{2} k^m E^m \]

*Harmonic gauge condition*

Number of independent solutions:
For each \( k^m \) satisfying \( k^n k^m = 0 \),

- \( E_{mn} \) - Symmetric 4x4 matrix
  - 10 components
  - harmonic gauge constraint: \(-4\)
  - remaining gauge freedom: \(-4\)

\[ 2 \text{ independent polarizations} \]

- Like \( E_{66} \)!

Example: Wave traveling in \( x^3 \)-direction.

\[ k^1 = k^2 = 0, \ k^3 = k^0 = K > 0. \]

Harmonic conditions:
\[ \{ k^1 E_{1} + k^0 E_{0} = k^2 E_{3} + k^0 E_{0} = 0 \}
\[ k^3 E_{3} + k^0 E_{3} = - (k^1 E_{30} + k^0 E_{00}) = \frac{1}{2} k^2 (E_{10} + E_{21} + E_{33} - E_{00}) \]

\[ k^3 = k^0 = K \Rightarrow \]
\[ \{ E_{1} + E_{0} = E_{3} + E_{0} = 0 \}
\[ E_{13} + E_{03} = -(E_{30} + E_{00}) = \frac{1}{2} (E_{11} + E_{22} + E_{33} - E_{00}) \]

\[ \Rightarrow \]
\[ E_{01} = -E_{31}, \quad E_{02} = -E_{32}, \quad E_{03} = -E_{33}, \quad E_{11}, \ E_{22}, \ E_{33}, \ E_{00} \]

\[ \text{dependent on other polarizations} \]
Residual gauge freedom: \( E_{13} \to E_{13} + k E_1 \)
\[ E_{23} \to E_{23} + k E_2 \]
\[ E_{33} \to E_{33} + 2k E_3 \]
\[ E_{00} \to E_{00} - 2k E_0 \]

\( E_{13} = E_{23} - E_{33} - E_{00} = 0 \)

\( \Rightarrow \) unphysical polarizations.

\( \Rightarrow \) Only two components \((E_{11}, E_{12})\) have independent physical significance.

\( E_{01} = -E_{31} = -E_{33} = 0 \)
\( E_{02} = -E_{32} = -E_{33} = 0 \)
\( E_{03} = -\frac{1}{2}(E_{11} + E_{00}) = 0 \)

from harmonic conditions and above gauge choice.

The polarization tensor in this gauge takes the form

\[
E_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & E^+ & E_x & 0 \\
0 & E_x & -E^+ & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}_{\mu\nu}
\]

for some \( E^+, E_x \)

Notice that in this gauge, \( K^\mu E_{\mu\nu} = 0 \) \( \Leftrightarrow \) transverse and \( E^\mu E_\mu = 0 \) \( \Leftrightarrow \) traceless

This is called transverse, traceless gauge.

**Caution:** Notice that we used the equations of motion with \( T_{\mu\nu} = 0 \) in order to deduce \( K_\mu K^\mu = 0 \), which led to \( K^3 = 0 \) here. If \( T_{\mu\nu} = 0 \), we might not be able to simultaneously satisfy the equations of motion \((I.e.\) the linearized Einstein eqs.) and the transverse + traceless conditions here.
Helicity of Gravitational Waves

Consider a rotation by angle $\theta$ about the $x^3-x^4$ axis,

$$A^\mu = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad (A^{-1})^\mu = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

The polar tensor tensor transforms as

$$E_{\mu \nu} \rightarrow E'_{\mu \nu} = (A^{-1})^\mu \nu (A^{-1})^\rho \sigma E_{\rho \sigma}$$

Defining

$$E_+ = E_{11} + i E_{12} = -E_{22} + i E_{12}$$
$$f_+ = E_{31} + i E_{32} = -E_{01} + i E_{02}$$

it is straightforward to check that under the rotation

$$E_+ = \exp(\pm 2i \theta) E_+ \iff \text{helicity } \pm 2$$
$$f_+ = \exp(\pm i \theta) f_+ \iff \text{helicity } \pm 1$$
$$E_{33} = E_{33}, \quad E'_{00} = E_{00} \iff \text{helicity } 0$$

Any plane wave which transforms as $\Psi' = e^{i h \theta} \Psi$ under a rotation by $\theta$ about the direction of motion is said to have helicity $h$.

In our analysis of plane wave solutions for $h = 0$, we chose $k^1 = k^2 = 0$, so motion is in the $x^3$ direction.

We also found that the physical components of $E_{\mu \nu}$ were $E_{11}$ and $E_{12}$, which could be replaced by the linear combination $E_+$.

Hence, gravitational waves are decomposed into helicity $\pm 2$, $\pm 1$, 0 parts, but only the helicity $\pm 2$ parts are physical.
As usual, comparison with E&M is useful.

Source-free Maxwell equations: \( \nabla \cdot F^{\mu \nu} = 0 \)
\( \nabla \times (\nabla \times A^\mu - \partial^\mu \phi) = 0 \)

In Lorenz gauge \( \nabla \cdot A^\mu = 0 \), \( \nabla \times F^{\mu \nu} = 0 \)

Gauge transformations \( A^\mu \rightarrow A^\mu + \partial^\mu \phi \) with \( \partial^\mu \partial_\mu \phi = 0 \) preserve the Lorenz gauge condition.

4 propagating degrees of freedom, \( 4-1-1 = 2 \)

Lorenz gauge condition
Residual gauge freedom

Under a rotation, \( A^\alpha \rightarrow (\Lambda^{-1})^\alpha \gamma_\mu \Lambda^\mu \)
One factor of \( \Lambda^{-1} \) (rather than two in transformations of \( h_{\mu \nu} \)) \( \rightarrow \) physical plane waves have helicity \( \pm 1 \) in electromagnetism.

(Exercise)

Motion of particles

To consider motion of particles in a background of \( h_{\mu \nu} \), we need to identify how \( h_{\mu \nu} \) appears in the metric tensor \( g_{\mu \nu} \). A natural guess is \( [g_{\mu \nu} = \delta_{\mu \nu} + h_{\mu \nu}] \) as in the Newtonian limit studied earlier.

Particles then follow the geodesic equation, with this gauge,

\[ \frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\nu \lambda} \frac{dx^\nu}{dt} \frac{dx^\lambda}{dt} = 0. \]
Motion of particles in a gravitational wave

Consider a particle initially at rest, \( \frac{dx^0}{dt} = 1, \frac{dx^i}{dt} = 0 \).

The particle follows the geodesic equation,

\[
\frac{d^2 x^m}{dt^2} + \Gamma^m_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0
\]

\[\Rightarrow \frac{d^2 x^m}{dt^2} + \Gamma^m_{00} \approx 0 \text{ near the initial instant} \]

\[
\Gamma^m_{00} = \frac{1}{2} g^{m i} \left( \partial_0 h_{0i} + \partial_i h_{00} - \partial_i h_{00} \right) + O(h^2)
\]

For a plane wave in transverse-traceless gauge, \( e_{0i} = e_{00} = 0 \).

\[\Rightarrow \Gamma^m_{00} \approx 0. \]

\[\frac{d^2 x^m}{dt^2} \approx 0. \quad \text{Particle at rest remains at rest or at least for short times.} \]

If a particle does not respond to a passing gravitational wave, then how would such a wave be detected?

Answer: Consider a collection of particles,

A physical gravitational wave would be in the form of a wavepacket, i.e., a superposition of plane waves.

For example, consider a superposition of plane waves in the \( x^3 \)-direction,

\[ k^m \sim (K, 0, 0, K) \]

\[ \eta_{\mu \nu} = \left( \frac{dK}{\tilde{K}(k)} e^{i k (x - t)} \tilde{E}_{\mu \nu}(k) + c.c. \right) \quad \text{complex conjugate} \]

Suppose \( \tilde{E}_{\mu \nu}(k) = E_{\mu \nu} \) independent of \( K \).

Then \( \eta_{\mu \nu} \) has the form \( \eta_{\mu \nu} \equiv f(z-x) E_{\mu \nu} + c.c. \)

\[ f(x) \rightarrow \]

\[ f(z-x) \rightarrow \]
In transverse traceless gauge,

\[ \varepsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon^{+} & \varepsilon^{x} & 0 \\ 0 & \varepsilon^{x} & -\varepsilon^{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

Suppose there are three particles in the x-y plane:

\[ x_1 = (0, 0, 0) \quad \text{Assume } \alpha \text{ is small} \]
\[ x_2 = (a, 0, 0) \quad \text{compared to the width of the wavepacket} \]
\[ x_3 = (10a, 0, 0) \]

The proper distance between these points is:

\[ (\Delta s_{xx})^2 = 9r_{xx} \alpha^2 = (1 + h_{xx}) \alpha^2 \]
\[ \Delta s_{xx} = \alpha \sqrt{1 + h_{xx}} \approx \alpha (1 + \frac{1}{2} (f(z-t)\varepsilon^+ + c.c.)) \]
\[ = \alpha (1 + \text{Re}(f(z-t)\varepsilon^+)) \]

\[ (\Delta s_{12})^2 = 9r_{12} \alpha^2 = (1 + h_{12}) \alpha^2 \]
\[ \Delta s_{12} = \alpha (1 - \frac{1}{2} (f(z-t)\varepsilon^+ + c.c.)) \]
\[ = \alpha (1 - \text{Re}(f(z-t)\varepsilon^+)) \]

As the distance between 1 and 3 shrinks, the distance between 1 and 2 grows, and vice versa.

A circular distribution of particles would be distorted into an ellipsoidal shape:

- \( f(z-t) > 0 \), \( E_+ < 0 \), \( E_x = 0 \) This distortion is the basis of gravitational wave searches.