Just as in electrodynamics, we can choose to fix a
gauge condition on $h_{\mu \nu}$ in order to make equations
look simpler.
For example, we may insist that $h_{\mu \nu}$ satisfy,
$$\mathcal{L}h^{\mu \nu} = \frac{i}{2} \partial_{\mu}h$$
Harmonic gauge.

To see that we can make this choice, assume
$$\mathcal{L}h^{\mu \nu} - \frac{i}{2} \partial_{\mu}h = f_{\mu}^\nu(x) \neq 0.$$ 
Let $h_{\mu \nu} \rightarrow h_{\mu \nu} + 2 \partial_{\mu} \xi_{\nu} + 2 \partial_{\nu} \xi_{\mu}$ (gauge transformation)
Then,
$$\mathcal{L}h^{\mu \nu} - \frac{i}{2} \partial_{\mu}h \rightarrow \mathcal{L}_{\nu}h^\mu_{\nu} - \frac{i}{2} \partial_{\mu}h + 2 \partial_{\mu} \xi_{\nu} + 2 \partial_{\nu} \xi_{\mu} - 2 \partial_{\nu} \xi_{\mu} = 2 \partial_{\nu} \xi_{\mu} \xi_{\nu}(x)$$
$$= 0 \text{ if } \partial_{\nu} \xi_{\mu} = - f_{\mu}^\nu(x),$$
which can be solved for $\xi_{\nu}(x)$.

Note that the harmonic gauge condition does not
completely fix the gauge, because we can make
a further gauge transformation with $\partial_{\nu} \xi_{\mu} = 0$
(4-th order equation) while maintaining the harmonic
gauge condition.

In the harmonic gauge, the linearized Einstein equations become:
$$\Box h_{\mu \nu} - \frac{i}{2} \partial_{\mu}h - \frac{i}{2} \partial_{\nu}h + \Box h + \Box h_{\mu \nu} = -2 \Box h$$
$$\text{or,}$$
$$\Box h_{\mu \nu} - \frac{i}{2} \Box h_{\mu \nu} = -2 T_{\mu \nu}$$
Linearized Einstein equations in harmonic gauge.
We can simplify the appearance of the linearized Einstein equations still further.

Given a 2-index tensor $A_{\mu\nu}$, define

$$\overline{A}_{\mu\nu} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) - \frac{1}{2} \gamma_{\mu\nu} A^0.$$

For a symmetric tensor like $h_{\mu\nu}$,

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} h$$

Also note, $\overline{h}_{\mu\nu} = h_{\mu\nu}$. (Exercise)

The linearized Einstein eqs. in harmonic gauge may be written

$$\delta \Box^0 \overline{h}_{\mu\nu} = -\lambda T_{\mu\nu}$$

Taking the trace:

$$\delta \Box^0 (h_{\mu}^{\mu} - \frac{1}{2} \delta_{\mu}^{\mu} h) = -\lambda T_{\mu}^{\mu}$$

or

$$\delta \Box^0 h = -\lambda T$$

Adding $\frac{1}{2} \gamma_{\mu\nu} \delta \Box^0 h$ to the linearized Einstein eqs. gives

$$\delta \Box^0 h_{\mu\nu} = -\lambda T_{\mu\nu} + \frac{1}{2} \gamma_{\mu\nu} \delta \Box^0 h = -\lambda T_{\mu\nu} + \frac{1}{2} \gamma_{\mu\nu} \lambda T$$

i.e.,

$$\delta \Box^0 h_{\mu\nu} = -\lambda \overline{T}_{\mu\nu}$$