Consider flat space with Cartesian coordinates \(s^a\),
\[
ds^2 = ds^1 ds^2 ds^3 ds^4 \quad \text{in ordinary space, or}
\[
ds^2 = ds^0 ds^1 ds^2 ds^3 \quad \text{in spacetime.}
\]

A constant vector field is one in which \(\frac{\partial V^\mu}{\partial x^a} = 0\).

(Note that in non-Cartesian coordinates this is not true.
\(\frac{\partial V^\mu}{\partial x^0} \neq 0\) in general.)

\[
\begin{array}{ccc}
\uparrow & \downarrow & \uparrow \\
\uparrow & \downarrow & \uparrow \\
\uparrow & \downarrow & \uparrow \\
\hline
V^x &=& 0 \\
V^y &=& 1 \\
V^z &=& \text{constant, but}
\end{array}
\]

\[
V^r = \sin \theta, \quad V^\theta = \cos \theta \quad \text{polar coords.}
\]

In curved space (time), a constant vector field satisfies \(\frac{\partial V^\mu}{\partial s^a} = 0\) in locally flat (Minkowski) coordinates.

General coordinates: at each point, \(V^0\) in \(x\)-coords.

Locally Minkowski:
\[
\frac{\partial V^\mu}{\partial s^a} = \frac{\partial x^0}{\partial s^a} \frac{\partial V^\mu}{\partial x^0} + \frac{\partial x^a}{\partial s^b} \frac{\partial V^\mu}{\partial x^b}
\]

\[
= \frac{\partial x^0}{\partial s^a} \left[ \frac{\partial V^\mu}{\partial x^0} + \nabla \frac{\partial V^\mu}{\partial x^0} \right]
\]

\[
= \frac{\partial x^0}{\partial s^a} \left[ \frac{\partial V^\mu}{\partial x^0} + V^0 \frac{\partial \partial V^\mu}{\partial x^0} \right]
\]

\[
\frac{\partial V^\mu}{\partial s^a} = \frac{\partial x^0}{\partial s^a} \frac{\partial V^\mu}{\partial x^0} V^0 \cdot \nu
\]
The condition for a vector field to be constant in curved space is $V^m;_r = 0$.

**Comment Derivable Along a Curve**

Locally Cartesian (inertial) coordinate $\xi^I(x)$

\[
\frac{DV^I}{d\lambda} = \lim_{\Delta \to 0} \frac{V^I(x+\Delta) - V^I(x)}{\Delta}
\]

\[
\frac{DV^I}{d\lambda} = \frac{\partial V^I}{\partial x^\mu} \frac{dx^\mu}{d\lambda} + \frac{\partial V^I}{\partial x^\sigma} \frac{dx^\sigma}{d\lambda} V^\sigma \cdot \nu
\]

\[
= \frac{\partial V^I}{\partial x^\sigma} \left( \frac{dx^\nu}{d\lambda} V^\sigma \cdot \nu \right)
\]

\[
\frac{DV^\sigma}{d\lambda} = \frac{dx^\nu}{d\lambda} \left( \frac{\partial V^\sigma}{\partial x^\nu} + \Gamma^\sigma_{\nu\mu} V^\mu \right)
\]

\[
\frac{dv^\sigma}{d\lambda} = \frac{dV^\sigma}{d\lambda} + \Gamma^\sigma_{\nu\mu} \frac{dx^\nu}{d\lambda} V^\mu
\]

Note that this last expression defines the component derivable along a curve even for vector fields defined only along the curve (like $X^m(t)$ describing the trajectory of a particle).
Parallel Transport of Vectors

Keep vector constant with respect to itself along a trajectory

Flat space: \( V^m(\gamma) = V^m(\gamma_0) \)

\[ \frac{DV^m}{DA} = 0 \rightarrow \text{Defines parallel transport} \]

\[ \frac{dV^m}{dx} = -\Gamma^m_{uv} \frac{dx^u}{dx} V^v \]

Parallel transport equation.

Along a geodesic the tangent vector \( V^m = \frac{dx^m}{d\lambda} \) is parallel transported.

\[ \frac{D}{DA} \left( \frac{dx^m}{d\lambda} \right) = 0 \rightarrow \frac{d^2x^m}{d\lambda^2} + \Gamma^m_{uv} \frac{dx^u}{d\lambda} \frac{dx^v}{d\lambda} = 0 \]

Geodesic Eqn.

Curvature

In curved spaces, parallel transport of a vector along a loop does not generally leave a vector invariant upon traversing a full cycle.

A vector does not return to itself.

Def: A manifold is flat if any vector parallelly transported along any closed loop returns the vector to itself.
Along path from \( A \) to \( B \): \( \frac{dV^\alpha}{d\tau} = 0 \)

\[
\frac{dV^\alpha}{d\tau} = -\Gamma^\alpha_{\mu 1} \frac{dx^\mu}{d\tau} V^\nu, \quad \frac{dV^\alpha}{d\tau} + \Gamma^\alpha_{\mu \nu} V^\nu = 0 \quad \text{along trajectory.}
\]

\[V^\alpha(B) = V^\alpha(A) + \int_{x^\alpha = b}^{x^\alpha = a} dx^1 (-\Gamma^\alpha_{\nu 1} V^\nu)\]

Similarly, \( V^\alpha(C) = V^\alpha(B) + \int_{x^\alpha = a}^{x^\alpha = c} dx^2 (-\Gamma^\alpha_{\nu 2} V^\nu)\)

\[V^\alpha(D) = V^\alpha(C) + \int_{x^\alpha = c}^{x^\alpha = d} dx^3 (\Gamma^\alpha_{\nu 3} V^\nu)\]

\[V^\alpha(A_{\text{return}}) = V^\alpha(D) + \int_{x^\alpha = d}^{x^\alpha = a} dx^1 (\Gamma^\alpha_{\nu 1} V^\nu)\]

\[V^\alpha(A_{\text{return}}) - V^\alpha(A) = \int_{x^\alpha = a}^{x^\alpha = a} dx^2 (\Gamma^\alpha_{\nu 2} V^\nu - \Gamma^\alpha_{\nu 1} V^\nu)\]

\[+ \int_{x^\alpha = a}^{x^\alpha = c} dx^3 (\Gamma^\alpha_{\nu 3} V^\nu - \Gamma^\alpha_{\nu 2} V^\nu)\]

\[= \frac{\delta a}{6} \left[ -\frac{\partial}{\partial x^1} (\Gamma^\alpha_{\nu 2} V^\nu) \right] + \frac{\delta c}{a} \left[ \frac{\partial}{\partial x^3} (\Gamma^\alpha_{\nu 1} V^\nu) \right] + \frac{\delta b}{b} \left[ -\frac{\partial}{\partial x^1} (\Gamma^\alpha_{\nu 2} V^\nu) \right] + \frac{\delta d}{d} \left[ \frac{\partial}{\partial x^3} (\Gamma^\alpha_{\nu 1} V^\nu) \right] \]
\[ V^\mu (A_{\text{return}}) - V^\mu (A) = \delta_v \delta_b \left\{ \left( \frac{\partial}{\partial x^1} \Gamma^\mu_{1\lambda} \right) V^\lambda - \Gamma^\lambda_{1\mu} \frac{\partial}{\partial x^1} V^\mu + \left( \frac{\partial}{\partial x^2} \Gamma^\mu_{2\lambda} \right) V^\lambda - \Gamma^\lambda_{2\mu} \frac{\partial}{\partial x^2} V^\mu \right\} \]

The vector \( V^\mu \) is parallel transported along the loop, so
\[ \frac{\partial V^\mu}{\partial x^1} = -\Gamma^\mu_{1\lambda} V^\lambda \quad \text{or} \quad \partial_{x^1} V^\mu = -\Gamma^\mu_{1\lambda} V^\lambda \]
along appropriate portions of the loop.

\[ \Rightarrow \quad V^\mu (A_{\text{return}}) - V^\mu (A) = \delta V^\mu \]

\[ \delta V^\mu = \delta_v \delta_b \left\{ \frac{\partial}{\partial x^1} \Gamma^\mu_{1\lambda} - \frac{\partial}{\partial x^1} \Gamma^\lambda_{1\mu} + \frac{\partial}{\partial x^2} \Gamma^\mu_{2\lambda} - \frac{\partial}{\partial x^2} \Gamma^\lambda_{2\mu} \right\} V^\lambda \]

\[ \delta V^\mu = \delta_v \delta_b R^\mu_{\lambda\sigma\rho} V^\sigma \]

More generally, if \( V^\mu \) is parallel transported around a loop spanning \( \delta_a \) in \( x^a \)-direction, \( \delta_b \) in \( x^b \)-direction, \( \delta_v \) in \( x^v \)-direction, \( \delta_x \) in \( x^x \)-direction:

\[ \delta V^\mu = \delta_v \delta_b \delta_a \delta_x \]

\[ R^\mu_{\lambda\sigma\rho} = \frac{\partial^2 \Gamma^\mu_{\sigma\rho}}{\partial x^\lambda \partial x^\mu} - \frac{\partial^2 \Gamma^\mu_{\lambda\rho}}{\partial x^\sigma \partial x^\mu} + \Gamma^\tau_{\lambda\sigma} \Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\lambda\rho} \Gamma^\mu_{\tau\sigma} \]

**Riemann Curvature Tensor**

Exercise: Show that \( R^\mu_{\lambda\sigma\rho} \) is a tensor.

A space (time) is flat iff. \( R^\mu_{\lambda\sigma\rho} = 0 \) everywhere.
Properties of $R^k_{\mu \nu \rho}$:

1) $R^k_{\mu \nu \rho}$ is the only tensor that can be constructed from $g_{\mu \nu}$ and its first and second derivatives.

2) $R^k_{\mu \nu \rho}$ can also be defined in terms of the commutator of covariant derivatives:

$$V^\alpha_{\nu \mu k} - V^\alpha_{\nu k \mu} = - V^\sigma_{\nu} R^\sigma_{\mu \nu k}$$

$$V^\lambda_{\nu k} - V^\lambda_{\nu k} = V^\sigma_{\nu} R^\lambda_{\mu \sigma \nu k}$$

3) Define $R^\sigma_{\mu \nu k} = 9 g^\sigma_{\mu} R^\sigma_{\nu \mu k}$

$$R^\sigma_{\mu \nu k} = \frac{1}{2} \left\{ \frac{\partial^2 g_{\nu \mu}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{\mu \nu}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{\mu \lambda}}{\partial x^k \partial x^\lambda} + \frac{\partial^2 g_{\nu \lambda}}{\partial x^k \partial x^\lambda} \right\}$$

$$+ 9 g_{\sigma} \left[ \Gamma^\sigma_{\nu \mu} \Gamma^\sigma_{k \lambda} - \Gamma^\sigma_{\nu k} \Gamma^\sigma_{\lambda \mu} \right]$$

4) $R^\mu_{\nu k} = R^k_{\nu \mu}$

$$R^\mu_{\nu k} = - R^\mu_{k \nu} = - R^\mu_{\mu \nu} = + R^\mu_{\nu \mu}$$

Algebraic Relations

$$R^\mu_{\nu k} + R^k_{\nu \mu} + R^\nu_{\mu k} = 0$$
Useful contractions of $R^i_{\mu \nu \kappa \lambda}$:

$$R_{\mu \kappa} = R^j_{\mu \nu \kappa} \quad \text{Ricci Tensor}$$

$$R = g^{\mu \nu} R_{\mu \nu} \quad \text{Curvature Scalar}$$

\underline{Bianchi Identities}

In a locally Cartesian (inertial) coordinate system, $\Gamma^i_{\mu \nu} = 0$, but $\partial^i \Gamma^i_{\mu \nu} \neq 0$.

$$R_{\mu \nu \kappa \lambda} = \frac{1}{2} \partial^i \left( \frac{\partial^j R_{\mu \nu \kappa \lambda}}{\partial x^j} - \frac{\partial^j R_{\mu \nu \kappa \lambda}}{\partial x^j} + \frac{\partial^j R_{\mu \nu \kappa \lambda}}{\partial x^j} + \frac{\partial^j R_{\mu \nu \kappa \lambda}}{\partial x^j} \right)$$

\underline{Exercise:}

$$R_{\mu \nu \kappa \lambda} + R_{\mu \nu \kappa \lambda} + R_{\mu \nu \kappa \lambda} = 0$$

cyclic permutation

This is a covariant relation, so it is true in arbitrary frames.

\underline{Contact with $g_{\mu \nu}$:}

$$R_{\mu \nu \kappa \lambda} - R_{\mu \nu \kappa \lambda} + R^\lambda_{\mu \nu \kappa} = 0$$

\underline{Contact with $g^\mu \nu$:}

$$R_{\mu \nu} - R^\lambda_{\mu \nu ; \lambda} - R^\lambda_{\mu \nu} = 0$$

$$\Rightarrow (R^\mu_{\nu} - \frac{1}{2} \delta^\mu_{\nu} R) ; \mu = 0$$

$$\Rightarrow (R^\mu_{\nu} - \frac{1}{2} g^\mu_{\nu} R) ; \mu = 0$$