Motion of a Gravitational Particle

We have already written equations describing the motion of freely falling particles in an arbitrary coordinate system:

\[
\frac{dx^u}{dt} + \Gamma^u_{v\lambda} \frac{dx^v}{dt} \frac{dx^\lambda}{dt} = 0
\]

where \(ds^2 = -g_{\mu\nu} dx^\mu dx^\nu\) is the (proper-time)^2, and \(\Gamma^u_{v\lambda} = \frac{1}{2} g^{u\rho}(\partial_v g_{\lambda \rho} + \partial_\lambda g_{v \rho} - \partial_\rho g_{v \lambda})\)

The metric \(g_{\mu\nu}\) and affine connection \(\Gamma^u_{v\lambda}\) contain information about the gravitational field.

We now have a theory for gravity in terms of the rank-2 tensor field \(h_{\mu\nu}\), which couples to matter through a term \(-\lambda h_{\mu\nu} T^{\mu\nu}\) in the Lagrangian density.

For a relativistic particle, the energy is \(mc^2 = m \frac{dE}{dt}\), with \(c=1\), \(E=m\gamma\).

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

The momentum is \(p^i = m \frac{dx^i}{dt} \gamma = m \frac{dx^i}{dt} \frac{dt}{d\tau} = m \frac{dx^i}{d\tau}\)

\[
P_i = \int_{\tau_0}^{\infty} \int_{x_0}^{x(t)} \frac{d^3x}{dt} \frac{d^3x}{d\tau} d\tau
\]

The term in the action \(-\lambda \int h_{\mu\nu} T^{\mu\nu} d^4x dt\) includes \(-\lambda \int h_{\mu\nu} \frac{d^4x}{dt} d^4x dt\) over \(x\).
\[-\gamma \text{Shou} \, T^{\mu\nu} \, d^3x \, dt = -\lambda \text{Shou} \, m \frac{d x^\mu}{d t} \frac{d x^\nu}{d t} \frac{d t}{d \tau} \]

By Lorentz invariance, the action should then contain:

\[-\lambda \text{Shou} \, T^{\mu\nu} \, d^3x \, dt = -\lambda \text{Shou} \, m \frac{d x^\mu}{d t} \frac{d x^\nu}{d \tau} \frac{d t}{d \tau} \]

where we identify:

\[ T^{\mu\nu} = m \frac{d x^\mu}{d t} \frac{d x^\nu}{d t} s^2(\tau - \tau(t)) \frac{d \tau}{d t} \]

In the absence of gravitation, the equations of motion:

\[ m \frac{d^2 x^\mu}{d t^2} = 0 \]

follows from an action:

\[ S_0 = \frac{1}{2} m \int d^4 x \, \frac{d x^\mu}{d t} \frac{d x^\nu}{d t} \, \gamma_{\mu\nu} \]

Adding to this the gravitational coupling above, we postulate the action describing particle motion in a (weak) gravitational field:

\[ S = \frac{1}{2} m \int d^4 x \, \frac{d x^\mu}{d t} \frac{d x^\nu}{d t} \, \gamma_{\mu\nu} - 2m \int d^4 x \, \gamma_{\mu
\nu} \frac{d x^\mu}{d t} \frac{d x^\nu}{d t} \]

\[ = \frac{1}{2} m \int d^4 x \, \frac{d x^\mu}{d t} \frac{d x^\nu}{d t} (\gamma_{\mu\nu} - 2\gamma_{\mu\nu}) \]

\[ = \frac{1}{2} m \int d^4 x \, \frac{d x^\mu}{d t} \frac{d x^\nu}{d t} \, g_{\mu\nu} \]

where \( g_{\mu\nu} = \gamma_{\mu\nu} - 2\gamma_{\mu\nu} \), \( \gamma_{\mu\nu} = \gamma_{\mu\nu} \)

The action is a functional of \( x^\mu(t) \) and \( \frac{d x^\mu}{d t} \).

Stationarizing the action gives the equations of motion:

\[ m \frac{d}{d t} (g_{\mu\nu} \frac{d x^\nu}{d t}) - \frac{1}{2} m \frac{d x^\mu}{d t} \frac{d x^\nu}{d t} \frac{d x^\nu}{d t} \frac{d x^\mu}{d t} = 0 \]
\[
\begin{align*}
&g_{\mu
u} \frac{d^2 x^\rho}{d t^2} + \left( \partial_\rho \left( g_{\mu
u} \right) \frac{d x^\rho}{d t} \right) \frac{d x^\rho}{d t} - \frac{1}{2} \left( \partial_\rho \left( g_{\mu
u} \right) \right) \frac{d x^\rho}{d t} \frac{d x^\rho}{d t} = 0 \\
&g_{\mu
u} \frac{d^2 x^\rho}{d t^2} + \frac{1}{2} \left( \partial_\rho \left( g_{\mu
u} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\nu\rho} \right) \right) \frac{d x^\rho}{d t} \frac{d x^\rho}{d t} = 0 \\
&\text{Symmetrizing in} \\
&\partial_\rho
\end{align*}
\]

Multiply by \( g^{\rho\alpha} \):

\[
\frac{d^2 x^\alpha}{d t^2} + \frac{1}{2} g^{\alpha\rho} (\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\nu\rho}) \frac{d x^\rho}{d t} \frac{d x^\rho}{d t} = 0
\]

\[
\Gamma_{\alpha\beta}^{\gamma}
\]

We have recovered the equation for a freely falling particle with affine connection \( \Gamma_{\alpha\beta}^{\gamma} \)!

\( \text{Note: In the present discussion, the metric is} \):

\[g_{\mu\nu} = \eta_{\mu\nu} - 2 \chi_{\mu\nu}.\]

The factor of \((-2\chi)\) is from the normalization of tensors in the particle action. Had we started with the free particle action in the absence of gravity also multiplied by \((-2\chi)\), then we would have defined \(g_{\mu\nu} = \eta_{\mu\nu} - \chi_{\mu\nu}\) as before.
Motion of particles in a gravitational wave

Suppose a pulse of gravitational radiation passes a collection of free particles.

Superposition of plane waves: \( \psi = (k, 0, 0, k) \)
\[
h_{\mu\nu} = \int d\mathbf{k} \tilde{f}(k) \epsilon(k) e^{ik(z-t)} \epsilon_{\mu\nu} + \text{c.c.}
\]

\[= f(z-t) \epsilon_{\mu\nu} + \text{c.c.} \]

By a gauge choice we can set
\[
\epsilon_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \epsilon_{x x} & \epsilon_{x y} & 0 \\
0 & \epsilon_{y x} & -\epsilon_{y y} & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Suppose there are three particles in the \( x-y \) plane:

\[X_1 = (0, 0, 0) \]
\[X_2 = (a, 0, 0) \]
\[X_3 = (0, a, 0) \]

Proper time defines physical distance between points:

\[\Delta t_{12}^2 = g_{xx} a^2 = (1-2\lambda h_{xx}) a^2 \]
\[\Delta t_{12} = a \sqrt{1-2\lambda h_{xx}} \approx a (1-\lambda h_{xx}) = a (1-\lambda \epsilon_{xx} f(z-t)) \]
\[\Delta t_{13}^2 = g_{yy} a^2 = (1-2\lambda h_{yy}) a^2 \]
\[\Delta t_{13} = a \sqrt{1-2\lambda h_{yy}} \approx a (1-\lambda h_{yy}) = a (1+\lambda \epsilon_{xx} f(z-t)) \]

As the distance between 1 and 3 shrinks, the distance between 1 and 2 grows and vice versa.
As the pulse of gravitational radiation passes, a circular distribution of particles is distorted into an ellipsoidal shape:

\[ f(t) \rightarrow \begin{cases} \lambda \varepsilon_{xx} < 0, & \varepsilon_{yy} = 0 \\ \lambda \varepsilon_{xx} = 0, & \lambda \varepsilon_{yy} < 0 \end{cases} \]

This distortion is the basis of gravitational wave searches.

(We will later have a less hand-wavy language to discuss the relative motion of nearby points, i.e. the geodesic deviation. We will not until our discussion of spacetime curvature to return to this.)