Review of Lagrangian Mechanics

Suppose a system is described by a set of $N$ coordinates $q_i(t)$, $i = 1, \ldots, N$.

**Action Principle:** There exists a functional of $q_i, \dot{q}_i,$ and $t$, called the action, which is stationary about variations $\delta q_i(t)$ along a classical path.

Lagrangian $L(q_i, \dot{q}_i, t)$

Action $S = \int_{t_1}^{t_2} dt \ L(q_i, \dot{q}_i, t)$

Variation $q_i(t) \rightarrow q_i(t) + \delta q_i(t)$

$\delta q_i(t_1) = \delta q_i(t_2) = 0$ (Boundary fixed)

$$\delta S = \int_{t_1}^{t_2} dt \ \left[ \sum_i \left( \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \dot{\delta q}_i \right) \right]$$

$$= \left[ \int_{t_1}^{t_2} dt \ \sum_i \left[ \frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \dot{\delta q}_i \right] \right]$$

$\delta S = 0$ for arbitrary small variation $\delta q_i(t)$ along a classical path

$\Rightarrow \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$ Euler-Lagrange Eq.
**Hamiltonian Formulation**

Given a Lagrangian $L$, define the canonical momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and $\dot{p}_i = \frac{\partial L}{\partial q_i}$ from Euler-Lagrange Eqs.

Define the Hamiltonian $H(p_i, q_i, t) = \sum_i p_i \dot{q}_i - L$

$H$ must be written in terms of $p_i$'s and $q_i$'s, not the $\dot{q}_i$'s, and the $p_i$'s and $q_i$'s must be independent. (This is not always possible!)

Vary the coordinates and momenta:

$$dH = \sum_i \left( \dot{p}_i \dot{q}_i + p_i \ddot{q}_i - \frac{\partial L}{\partial q_i} \dot{q}_i - \frac{\partial L}{\partial \dot{q}_i} \right)$$

$$= \sum_i \left( \dot{q}_i \dot{p}_i - p_i \ddot{q}_i \right) \quad \text{using the Euler-Lagrange Eqs.}$$

$$= \sum_i \left( \frac{\partial H}{\partial \dot{p}_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i \right)$$

$$\Rightarrow \frac{\partial H}{\partial \dot{p}_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i \quad \text{Hamilton's Eqs.}$$

If $\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = \sum_i \left( \frac{\partial H}{\partial \dot{p}_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i \right) - \frac{\partial L}{\partial t}$

$$= \sum_i \left( \dot{q}_i \dot{p}_i - p_i \ddot{q}_i \right) = 0$$

In that case $H$ is called the energy and is conserved.
Symmetries and Conservation Laws

Noether's Theorem: For every global symmetry parametrized by a continuous parameter, there is a corresponding conservation law.

Under a variation of $q_i(t)$,

$$L \rightarrow L + \frac{\lambda}{2} \left( \frac{\partial L}{\partial \dot{q}_i} \delta q_i + \frac{\partial L}{\partial q_i} \delta \dot{q}_i \right)$$

$$= L + \frac{\lambda}{2} \left( \dot{p}_i \delta q_i + p_i \delta \dot{q}_i \right) \quad \text{using the E-L Eqs.}$$

$$= L + \frac{\lambda}{2} \frac{d}{dt} \left( \sum p_i \delta q_i \right) \quad \text{using } \delta \dot{q}_i = \frac{d}{dt} \delta q_i.$$

Now consider a class of variations parametrized by a continuous parameter $\lambda$: $\delta q_i(t, \lambda) = \lambda \frac{\partial q_i}{\partial \lambda} \bigg|_{\lambda=0}$

Suppose under this class of transformations

$L \rightarrow L + \lambda \frac{dF}{dt}$ for some $F(q_i, \dot{q}_i, t)$, $F(t)$ - $F(t_0) = 0$.

(not using the Eqs. of motion)

Then the class of transformations $q_i(t) \rightarrow q_i(t', \lambda)$ is called a symmetry.

Hamilton's principle: $0 = S - \int_{t_1}^{t_2} dt \delta L = \int_{t_1}^{t_2} dt \lambda \frac{dF}{dt}$

$$= \lambda \left( F(t_2) - F(t_1) \right) = 0$$
Such a symmetry transformation does not change the action or the equations of motion.

Around the eqs. of motion $L \to L + \frac{d}{dt} \left( \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)$

$$= L + \lambda \frac{d}{dt} \left( \sum_i \frac{\partial F}{\partial \dot{q}_i} \right) \bigg|_{\lambda=0} \quad \text{for } \lambda \ll 1$$

$$= L + \lambda \frac{dF}{dt} \quad \text{by assumption}$$

$$\Rightarrow \frac{d}{dt} \left( \sum_i \frac{\partial \mathbf{P}_i}{\partial \dot{q}_i} \right) \bigg|_{\lambda=0} - F(q_i, \dot{q}_i, t) = 0$$

The quantity $Q = \sum_i \frac{\partial \mathbf{P}_i}{\partial \dot{q}_i} \bigg|_{\lambda=0} - F(q_i, \dot{q}_i, t)$ is conserved.

Example: Space translation of point particles

Lagrangian $L = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 - \sum_{ij} V_{ij} (|\mathbf{r}_i - \mathbf{r}_j|)$

$\mathbf{r}_i \to \mathbf{r}_i + \alpha \lambda$ takes $L \to L$ for any fixed $\alpha$.

$$\frac{\partial L}{\partial \dot{r}_i} = m_i \ddot{r}_i, \quad F = 0, \quad \ddot{r}_i = \frac{\partial L}{\partial \dot{r}_i} = m_i \ddot{r}_i$$

$Q = \sum_i \dot{r}_i \cdot \ddot{r}_i$ is conserved. Since this is true for all $\lambda$,

$\sum_i \dot{r}_i$ is conserved.

Translation invariance $\Rightarrow$ Momentum Conservation.
Example: Time translations $\tilde{\mathbf{x}}_i(t) \rightarrow \tilde{\mathbf{x}}_i(t+\tau)$

$$\left. \frac{\partial \tilde{\mathbf{x}}_i(t, \tau)}{\partial \tau} \right|_{\tau=0} = \frac{d\tilde{\mathbf{x}}_i(t)}{dt}$$

$$\delta L = \gamma \frac{dL}{dt} \quad \Rightarrow \quad F = L$$

$$Q = \sum_i \Pi_i \dot{\mathbf{x}}_i - L = H$$ is conserved.

*Time translation invariance → Energy conservation*
Classical Field Theory

\( \phi_i(t) \mapsto \phi_i(x, t) \) infinite set of generalized coordinates

\( t \mapsto t \)

\( i \mapsto i, \dot{x} \)

\( \Sigma_i \mapsto \Sigma_i \int d^3 \vec{x} \)

Lagrangian \( L(\phi_i, \dot{\phi}_i, t) \) \( \rightarrow \) Lagrangian density \( L(\phi_i, \partial_\mu \phi_i, x^\mu) \)

\( L = \frac{1}{2} \int d^3 x L \) integrated over a spacelike time slice.

We assume that \( L \) is local in space and time, and depends on at most first derivatives w.r.t. \( t, \dot{x} \).

If \( L \) is a Lorentz scalar, then the Euler-Lagrange eqs will be Lorentz covariant.

\[
\delta S = \frac{1}{2} \int \left[ \frac{\partial L}{\partial \phi_i} \delta \phi_i + \frac{\partial L}{\partial (\partial_\mu \phi_i)} \delta (\partial_\mu \phi_i) \right] d^3 x 
= \int \Pi_i \delta \phi_i
\]

under arbitrary variations \( \delta \phi_i \) such that \( \delta \phi_i(t_1, \vec{x}) = \delta \phi_i(t_2, \vec{x}) = 0 \).

Integrate by parts \( \rightarrow \delta S = \frac{1}{2} \int \left[ \frac{\partial L}{\partial \phi_i} - \partial_\mu \Pi_i^\mu \right] \delta \phi_i = 0. \)

\[
\Rightarrow \frac{\partial L}{\partial \phi_i} - \partial_\mu \Pi_i^\mu = \left[ \frac{\partial L}{\partial \phi_i} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi_i)} \right] = 0
\]

Euler–Lagrange Eq
The canonical momentum is \( \Pi_i(t, x) = \Pi^0_i(t, x) \).

(Don't think of \( \Pi^a_i \) as a 9-vector generalization of the canonical momentum.)

Hamiltonian:
\[
H = \int d^4x \left[ \Pi_i \partial_0 \phi_i - L \right]
\]

Hamiltonian density:
\[
H = \left[ \sum_i \Pi_i \partial_0 \phi_i - L \right]
\]

**Example:** Most general \( L \) satisfying:

1. \( L \) is a Lorentz scalar.
2. \( L \) built from one real scalar field \( \phi = \phi^* \).
3. \( L \) quadratic in \( \partial_\alpha \phi \).

For linear eps. of motion,

\[
L = \frac{1}{2} a \left[ \partial_\alpha \phi \partial^\alpha \phi + b \phi^2 \right]
\]

We can rescale \( \phi \rightarrow \frac{\phi}{\sqrt{a}} \), \( L \rightarrow \frac{1}{2} \left[ \partial_\alpha \phi \partial^\alpha \phi + b \phi^2 \right] \)

\[
\Pi^\mu = \frac{\partial \Sigma}{\partial (\partial_\mu \phi)} = \partial_\mu \phi \left( \frac{1}{2} a \phi \partial^\alpha \phi \phi + \frac{1}{2} b \phi^2 \right)
\]

Use \( \frac{\partial (\partial_\alpha \phi)}{\partial (\partial_\mu \phi)} = \delta_\mu^\alpha \)

\[
\Pi^\mu = \pm \frac{1}{2} \phi \partial_\lambda \phi \left( \delta_\mu^\lambda (\partial_\alpha \phi) + (\partial_\alpha \phi) \delta_\mu^\lambda \right)
\]

\[
= \pm \frac{1}{2} (\phi \partial_\alpha \phi \phi + \gamma_\mu^\alpha (\partial_\alpha \phi))
\]

\[
= \pm \partial_\alpha \phi
\]

\[
\partial_\mu \Pi^\mu - \frac{\partial L}{\partial \phi} = 0 \Rightarrow \pm \left( \partial_\alpha \partial^\alpha \phi - b \phi \right) = 0
\]
\[ H = \prod t^a \phi - L \]
\[ = (\pm \bar{\phi} \phi)(\bar{\phi} \phi) + \frac{1}{2} \left[ (\bar{\phi} \phi)^2 (\bar{\phi} \phi) + (\bar{\phi} \phi)^2 (\bar{\phi} \phi) + b \phi^2 \right] \]
\[ = \pm \frac{1}{2} (\bar{\phi} \phi)^2 (\bar{\phi} \phi) + \frac{1}{2} (\bar{\phi} \phi)^2 (\bar{\phi} \phi) + \frac{1}{2} b \phi^2 \]
\[ = \pm \frac{1}{2} (\bar{\phi} \phi)^2 + \frac{1}{2} (\bar{\phi} \phi)^2 + \frac{1}{2} b \phi^2 \]

It is possible semi-definite if:
\[ L \text{ can be zero} \]

1) choose the -ve sign in L,
2) \( b > 0 \).

\[ L = -\frac{1}{2} (\bar{\phi} \phi)^2 (\bar{\phi} \phi) - \frac{1}{2} m^2 \phi^2 \]
More Classical Field Theory

Example: Two real scalar fields $\phi_1(x), \phi_2(x)$.

Most general Lagrangian density satisfying:

1) $L$ is a Lorentz scalar
2) $L$ is quadratic in $\phi_1, \phi_2, \partial_m \phi_1, \partial_m \phi_2$
3) Symmetry under $\phi_1 \to \phi_1 \cos \theta + \phi_2 \sin \theta$
   $\phi_2 \to -\phi_1 \sin \theta + \phi_2 \cos \theta$

(like rotation in two dimensions)

$$L = \frac{1}{2} \sum_{i=1}^{2} \left( \partial_m \phi_i \partial^m \phi_i - m^2 \phi_i^2 \right)$$

Arbitrary convention so that Hamiltonian is bounded below

Euler-Lagrange Equations:

$$\partial_m \left( \frac{\partial L}{\partial \partial_m \phi_i} \right) = \frac{\partial L}{\partial \phi_i}$$

$$\partial_m \partial^m \phi_i = + m^2 \phi_i$$

Conserved current:

$$\frac{\partial \phi_1}{\partial \theta} \bigg|_{\theta=0} = \phi_2$$

$$\frac{\partial \phi_2}{\partial \theta} \bigg|_{\theta=0} = -\phi_1$$

Infinitesimal symmetry transformation:

$$\phi_1 \to \phi_1 + \theta \phi_2, \quad \phi_2 \to \phi_2 - \theta \phi_1$$
\[ S \rightarrow S + \int \! d^4 x \sum_i \left( \frac{\partial L}{\partial \delta \phi_i} \delta \phi_i + \frac{\partial L}{\partial (\partial_\mu \phi_i)} \partial_\mu \delta \phi_i \right) \]

\[ = S + \int \! d^4 x \sum_i \left[ \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi_i)} \right) \delta \phi_i + \frac{\partial L}{\partial (\partial_\mu \phi_i)} \partial_\mu \delta \phi_i \right] \]

(Using the E-L equations)

\[ = S + \int \! d^4 x \sum_i \partial_\mu \left[ \left. \frac{\partial L}{\partial (\partial_\mu \phi_i)} \right|_{\partial_\mu=0} \right] \delta \phi_i \]

(Using \( \delta \phi_i = \theta \frac{\partial \phi_i}{\partial \theta} \mid_{\theta=0} \))

But under the symmetry transformation \( L \rightarrow L' \), \( S \rightarrow S' \).

So \( \delta_m J^m = 0 \), where \( J^m = \sum_i \frac{\partial L}{\partial (\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial \theta} \mid_{\theta=0} \)

More generally, a symmetry leaves \( S \) invariant, for infinitesimal symmetry parameters \( \theta \). Let \( L \rightarrow L + \theta \partial \Xi^m \) for some \( \Xi^m(\phi_i(x), \partial \phi_i(x), x) \)

W/O using the E-L equations.

Then \( J^m = \sum_i \frac{\partial L}{\partial (\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial \theta} \mid_{\theta=0} - \Xi^m \)

For our theory, \( J^m = - (\partial_\mu \phi_1) \phi_2 + (\partial_\mu \phi_2) \phi_1 \)
Interpretation of current conservation:

\[ \partial_m J^m = 0 \]

\[ \int d^3x \partial_0 J^0 = -\int d^3x \partial_i J^i \]

\[ = -\int d^2x \frac{n^i J^i}{\partial V} \]

\[ n^i = \text{unit normal to boundary of } V \]

Hence, \( \frac{d}{dt} \int d^3x J^0 = -\int d^2x n^i J^i \)

Define charge \( Q = \int d^3x J^0 \)

Then \( \frac{dQ}{dt} = -\int d^2x \frac{n^i J^i}{\partial V} \)

\( \text{The rate of change of charge in a volume } V \text{ is the flux of current into } V \)

\[ \text{In our example, } Q = \int d^3x (\Phi_2 \partial_0 \Phi_1 + \Phi_1 \partial_0 \Phi_2) \]
The Energy-Momentum Tensor

As an application of Noether's theorem in field theory, consider a theory of a scalar field \( \phi(x) \) invariant under spacetime translations.

\[ \phi(x) \rightarrow \phi(x + a) = \phi(x) + a^m \partial_m \phi(x) \text{ for infinitesimal } a^m. \]

Gyration density \( J(\phi, \partial \phi) \) transforms similarly:

\[ J \rightarrow J + a^m \partial_m J = J + a^\nu \partial_\mu (J^\mu \partial_\nu J) \]

For each \( \nu = 0, 1, 2, 3 \) there is a conserved current:

\[ T^\mu_\nu = -\frac{\partial J}{\partial (\partial^\mu \phi)} \partial_\nu \phi + J^\mu \partial_\nu J \]

Energy-Momentum Tensor

Conserved charge associated with translations:

\[ H = \int d^4x \ T^{00} = \int d^3x \left[ \frac{\partial J}{\partial (\partial^0 \phi)} \partial_0 \phi - J^0 \right] \]

\[ T^0_\mu = \text{canonical momentum} \]

Conserved charge associated with spatial translations:

\[ P^\mu = \int d^3x \ T^{0\mu} = \int d^3x \ T^{0 i} = -\int d^3x \left[ \frac{\partial J}{\partial (\partial^0 \phi)} \partial_i \phi \right] = -\int d^3x \ T^{0 i} \partial_i \phi \]

\[ P^\mu = \int d^3x \ T^{i\nu} \partial_\nu \phi_\mu \]

Spacial Momentum
Symmetry of the Energy-Momentum Tensor

In an arbitrary Lorentz-invariant theory, the energy-momentum tensor can be made symmetric in exchange of its curls.

Given fields \( \phi_a \), \( T^{\mu \nu} = \sum_a \frac{\partial L}{\partial (\partial_a \phi_a)} \partial_a \phi_a - \eta^{\mu \nu} \)

Add to \( T^{\mu \nu} \) a tensor \( \Delta T^{\mu \nu} = \partial_a A^{\mu \nu} \) for some \( A^{\mu \nu} \), antisymmetric in \( \tau \to \eta \).

\[ \delta \int (T^{\mu \nu} + \Delta T^{\mu \nu}) = \partial_a T^{\mu \nu} + \partial_\nu \Delta A^{\mu \nu} \]

\[ \approx \partial_\tau T^{\mu \nu} \] by symmetry of mixed partial derivatives.

Hence, \( T^{\mu \nu} + \Delta T^{\mu \nu} \) is conserved if \( T^{\mu \nu} \) is conserved.

Under an infinitesimal Lorentz transformation suppose \( \phi_a \to \phi'_a = \sum_b \phi_b (U^{-1}) \)

\( \sum_{b} \xi_b^{\mu \nu} e_{a b} \Sigma^{\mu \nu}_{a b} \) is a matrix rep. of the Lorentz group.

\( \sum_{a b} \xi_b^{\mu \nu} e_{a b} \Sigma^{\mu \nu}_{a b} \) is a fundamental, antisymmetric, traceless tensor \( \Lambda^{\mu \nu} \sim S^{\mu \nu} - \eta^{\mu \nu} \)

Locally invariance of \( L \) implies
\[ \frac{\partial L}{\partial \phi_a} \delta \phi_a + \frac{\partial L}{\partial (\partial_a \phi_a)} \delta (\partial_a \phi_a) = 0 \]

where \( \delta \phi_a = \sum \xi_b^{\mu \nu} e_{a b} \Sigma^{\mu \nu}_{a b} \phi_b \)

Using E-L eqn,
\[ \sum_{a} \left[ \left( \partial_a \Pi^{\mu a} \right) \delta \phi_a + \Pi^{\mu a} \delta (\partial_a \phi_a) \right] = 0 \]
where \( \Pi^a_\mu = \frac{\partial L}{\partial (\partial_\mu \phi_a)} \).

Use \( \delta(\partial_\mu \phi_a(\mu)) = \partial_\mu \delta \phi_a - \epsilon^\nu_\mu \partial_\nu \phi_a + \theta(\epsilon^2) \)

\[ \sum_a \left[ \partial_\mu (\Pi^a_\mu \delta \phi_a) - \Pi^a_\mu \gamma^{\mu\lambda} \epsilon_{\lambda\mu} \partial_\nu \phi_a \right] = 0 \]

\[ \sum_{a_6} \partial_\lambda \left( \Pi^a_\mu \epsilon_{\mu \nu} \Sigma_{a_6} \phi_a \right) = \sum_a \Pi^a_\mu \epsilon_{\lambda \mu} \partial_\nu \phi_a \]

(\ast) \[ \sum_a \frac{1}{2} \left( \Pi^a_\mu \partial_\nu \phi_a - \Pi^a_\nu \partial_\mu \phi_a \right) = \sum_{a_6} \partial_\lambda \left( \Pi^a_\mu \Sigma_{a_6} \phi_a \right) \]

Define \( A^{\mu\nu} = \frac{1}{2} \sum_{a_6} \left( \Pi^a_\mu \Sigma_{a_6} \phi_a - \Pi^a_\nu \Sigma_{a_6} \phi_a \right) \phi_a \)

\[ T^{\mu\nu} = T^{\mu\nu} + \gamma^{\mu\nu} A^{\lambda\mu} \]

\[ = \sum_a \Pi^a_\mu \partial_\nu \phi_a - \gamma^{\mu\nu} L \]

\[ + \sum_{a_6} \partial_\lambda \left( \Pi^a_\mu \Sigma_{a_6} \phi_a \right) - \sum_{a_6} \partial_\lambda \left( \Pi^a_\mu \Sigma_{a_6} + \Pi^a_\nu \Sigma_{a_6} \phi_a \right) \phi_a \]

\[ = \frac{1}{2} \sum_a \left( \Pi^a_\mu \partial_\nu \phi_a + \Pi^a_\nu \partial_\mu \phi_a \right) - \gamma^{\mu\nu} L \]

\[ - \sum_{a_6} \partial_\lambda \left( \left( \Pi^a_\mu \Sigma_{a_6} + \Pi^a_\nu \Sigma_{a_6} \right) \phi_a \right) = \]

which is symmetric in \( \mu \leftrightarrow \nu \) and conserved.