Coordinate Singularities

In the Schwarzschild spacetime curvature
invariants like $R_{μνρσ}R^{μνρσ}$ behave regularly
around $r = 2GM$, but the metric depends on
$(1 - 2GM/c^2)^{1/2}$ which behaves singularly.

This is an example of a coordinate singularity
which can be eliminated by a change of coordinates.
A simple analogy is provided by the metric

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2, \quad -\infty < x < \infty, \quad 0 < t < \infty$$

A change of coordinates $t \to t' = 1/t$ removes
the singularity at $t = 0$:

$$ds^2 = (t')^4 \left(-\frac{dt}{t^2}\right)^2 + dx^2$$

$$= -(dt')^2 + dx^2 \quad \text{Minkowski space.}$$

The region covered by the original coordinates $0 < t < \infty$
is the upper-half plane in Minkowski space,
$0 < t' < \infty$.

The spacetime described by the original metric is geodesically
complete as $t \to 0$, meaning all geodesics approaching $t = 0$
extend to arbitrary values of their affine parameter $\lambda$. 
However, geodesics may reach $t = \infty$ for finite values of their affine parameter, so the coordinates $(x, t)$ do not describe a geodesically complete spacetime.

On the other hand, the geometry described by $(x, t)$ may be made geodesically complete by extending the coordinate range from $\mathbb{R} \times (-\infty, \infty)$ to $\mathbb{R} \times (-\infty, \infty)$.

Another example, similar in some respects to the Schwarzschild case, is the Rindler Spacetime,

$$d s^2 = -x^2 \, dt^2 + d x^2, \begin{cases} -\infty < t < 0, \quad 0 < x < \infty \\ 0 < t < \infty, \quad -\infty < x < 0 \end{cases}$$

The metric appears singular at $x = 0$. Geodesics terminate with finite affine parameter at $x = 0$, but the curvature is regular as $x \to 0$. Indeed, $R_{\mu \nu \lambda \sigma} = 0$ everywhere in the spacetime.

Null geodesics:

$$-x^2 \left( \frac{dt}{d\tau} \right)^2 + \left( \frac{dx}{d\tau} \right)^2 = 0$$

$$\left( \frac{dt}{d\tau} \right)^2 = \frac{1}{x^2}$$

$$t = \pm \ln x + \text{const.}$$

$\uparrow$ $+$ = "outward" \quad $\downarrow$ $-$ = "inward"

Define coordinates

$\nu = t - \ln x$, \quad -\infty < \nu < 0$

$\psi = t + \ln x$, \quad 0 < \psi < \infty$
\[ v-u = 2 \ln x \]
\[ x^2 = e^{v-u} \]
\[ v+u = 2t \]
\[ t = \frac{v+u}{2} \]

\[ dx = \frac{1}{2} e^{\frac{v+u}{2}} dv - \frac{i}{2} e^{\frac{v-u}{2}} du \]
\[ dt = \frac{1}{2} dv + \frac{i}{2} du \]

\[ ds^2 = -\pi^2 dt^2 + dx^2 = -\frac{e^{v-u}}{4} \left[ (dv^2 + du^2 + 2dudv) + \frac{e^{v-u}}{4} \left( dv^2 + du^2 - 2dudv \right) \right] \]

\[ ds^2 = -e^{v-u} dudv \]

\[ \left\{ \begin{array}{l}
-\infty < u < \infty \\
-\infty < v < \infty 
\end{array} \right. \]

**Metric Indep. of t -> Killing vector \( \xi^m = \delta^m_t \).**
**Constant of Trajectory:** \( \hat{E} = -9 \int t \frac{dx^m}{dt} \)

\[ = x^2 \frac{dt}{dt} \]
\[ = e^{v-u} \left( \frac{1}{2} \frac{dv}{dt} + \frac{i}{2} \frac{du}{dt} \right) \]

\[ u = \text{constant} : \text{"outgoing" trajectory} \]

\[ \text{Const.} = \frac{1}{2 \hat{E}} \int e^{v-u} dv \]
\[ = \text{Const.} + \left( \frac{e^{-u}}{2 \hat{E}} \right) e^v \]
\[ \approx \text{const.} \]
$V = \text{const} : "\text{ingoing" geodesic}$

$$T'_{in} = \frac{1}{\epsilon^2} \int e^{V-U} \, du = \text{const} - \frac{e^V}{\epsilon^2} e^{-u}$$

$T'_{in}$ and $T'_{out}$ not take a new choice of coordinates:

$$U = -e^{-u}, \quad V = e^v \quad \begin{cases} \infty < U < 0 \\ 0 < V < \infty \end{cases}$$

$$\Rightarrow ds^2 = -dU \, dV$$

"null geodesic coordinates"$

By extending the coordinate range to $\begin{cases} \infty < U < 0 \\ -\infty < V < 0 \end{cases}$

the spacetime becomes geodesically complete.

To show that the geometry extended in this way is just Minkowski space, define

$$T = \frac{U + V}{2}, \quad X = \frac{V - U}{2}$$

$$\Rightarrow ds^2 = -dT^2 + dX^2 \quad \begin{cases} -\infty < T < \infty \\ -\infty < X < \infty \end{cases}$$

In terms of the original coordinates,

$$x = (X^2 - T^2)^{1/2}$$

$$t = \tanh^{-1} \left( \frac{T}{X} \right)$$
Poincaré spacetime is the region I ($x > 1/\mu$) of Minkowski spacetime.

Consider a (non-geodesic) trajectory $x = \text{const.}$ in the original coordinates.

The proper acceleration is

$$a^m = \frac{D}{d\tau} \left( \frac{d x^m}{d\tau} \right) = U^m U^\nu \Gamma^\nu_{\mu\nu}$$

and $U^m U^m \Gamma_{\mu\nu} = -1 \Rightarrow U^m = \left( \frac{1}{\mu}, 0 \right)$.

The nonvanishing Christoffel symbols are

$$\Gamma^t_{\tau t} = \Gamma^t_{tt} = \frac{1}{x^2}$$

$$\Gamma^\tau_{\tau t} = \Gamma^\tau_{tt} = \frac{1}{x}$$

$$a^m = U^\nu \left( \frac{d}{d\tau} U^m + \Gamma^m_{\nu\lambda} U^\nu U^\lambda \right)$$

$$= \left( \frac{1}{u^2} \right) \Gamma^m_{tt} = \frac{1}{x^2} \Gamma^m_{tt}$$

$$\Gamma^\tau_{\tau t} = \frac{1}{x}, \quad a^\tau = 0$$

-> The Poincaré coordinates $(x, t)$ describe Minkowski space in an accelerated coordinate system.
Kruskal Coordinates

Consider the \( r, t \) part of the Schwarzschild metric:
\[
ds^2 = -\left(1 - \frac{2\,M}{r}\right) dt^2 + \left(1 - \frac{2\,M}{r}\right)^{-1} dr^2\]

Null geodesics satisfy
\[
-\left(1 - \frac{2\,M}{r}\right) \left(\frac{dt}{dr}\right)^2 + \left(1 - \frac{2\,M}{r}\right)^{-1} \left(\frac{dr}{dt}\right)^2 = 0
\]
\[
\Rightarrow \left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{2\,M}{r}\right)^{-2}
\]

Solutions:
\[
t = \pm r_* + \text{constant},
\]
where
\[
r_* = r + 2\,M \ln \left(\frac{r}{2\,M} - 1\right)
\]
is the "Regge - Wheeler - tortoise coordinate."

Note that
\[
\frac{dr_*}{dr} = \left(1 - \frac{2\,M}{r}\right)^{-1}.
\]

Define null coordinates \( u, v \):
\[
u = t - r_*
\]
\[
v = t + r_*
\]
\[
\Rightarrow ds^2 = -\left(1 - \frac{2\,M}{r}\right) du \, dv, \quad \text{where } r = r(u, v)
\]
from
\[
r_* = \frac{v-u}{2}
\]
\[
r + 2\,M \ln \left(\frac{r}{2\,M} - 1\right) = \frac{v-u}{2}
\]
\[
\frac{r}{2\,M} - 1 = e^{\frac{v-u}{4\,M}} e^{-\frac{r}{2\,M}}
\]
\[
\Rightarrow r = \frac{v-u}{2\,M} (1 - \frac{2\,M}{r})
\]
\[ ds^2 = -\frac{2GM}{r} e^{-r/2am} \left( -u/v + v/4am \right) \ du \ dv \]

Non-singular as \( r \to \infty \), \( (u \to 0 \ or \ v \to 0) \)

Define new coordinates \( U = -\frac{u}{v/4am} \)
\( V = e^{v/4am} \)

\[ ds^2 = -\frac{32G^2m^3}{r} e^{-r/2am} \ du \ dv \]

No singularity at \( r=2am \) \( (U=0 \ or \ V=0) \)

Change coordinates to \( T = \frac{U+V}{2} \), \( X = \frac{V-U}{2} \) and \( r(T, X) \)

\[ ds^2 = \frac{32G^2m^3}{r(\ln u, v)} e^{-r/2am} \left( -dT^2 + dX^2 + r^2 d\Omega^2 \right) \]

Schrödinger metric in Kruskal coordinates

\[ \left( \frac{r}{2am} - 1 \right) e^{-r/2am} = X^2 - T^2 \]
\[ \frac{t}{2am} - \ln \left( \frac{T+X}{X-T} \right) = 2 \cosh^{-1} \left( \frac{T}{X} \right) \]

The singularity at \( r=0 \) is a true curvature singularity, \( \mathbf{R}_{\mu \nu} \mathbf{R}_{\mu \nu} \to \infty \) as \( r \to 0 \). The allowed range of coordinates \( X, T \) follows from the condition \( r > 0 \)

\[ X^2 - T^2 > -1 \]
Region I: $r > 2GM$ of original Schwarzschild spacetime.

Region II: Black Hole - all observers in this region reach the singularity at $r=0$ in finite proper time.

Region III: White Hole - all observers originated at $r=0$ ($x = -(r^2-1)^{1/2}$) and leave region III in finite proper time.

Region IV: looks like Region I - asymptotically flat, $r > 2GM$. 

The curve remains a 2-sphere.