Killing Vector Fields

In order to describe motion in the Schwarzschild spacetime it will be useful to identify certain constants of the motion. We first digress with a more general discussion.

Consider a spacetime with metric $g_{\mu \nu}$. Under an infinitesimal coordinate transformation $x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu$,

$$g'_{\mu \nu}(x') = \frac{\partial x^k}{\partial x'^\mu} \frac{\partial x^l}{\partial x'^\nu} g_{kl}(x(x'))$$

$$= \left( \delta^\mu_\nu + \frac{\partial \xi^\mu}{\partial x'^\mu} \right) \left( \delta^\mu_\nu + \frac{\partial \xi^\nu}{\partial x'^\nu} \right) g_{\mu \nu}(x) + \frac{\partial \xi^\mu}{\partial x'^\mu} \theta_{\mu \nu} + \theta(\xi^2)$$

$$= g_{\mu \nu}(x') + g_{\mu \nu}(x) \frac{\partial \xi^\mu}{\partial x'^\mu} + g_{\nu \lambda}(x) \frac{\partial \xi^\lambda}{\partial x'^\nu} + \frac{\partial \xi^\mu}{\partial x'^\mu} \theta_{\mu \nu} + \theta(\xi^2)$$

$$= g_{\mu \nu}(x') + \left[ \frac{\partial \xi^\mu}{\partial x'^\mu} (g_{\nu \lambda} \delta^\lambda_\mu) - g_{\lambda \mu} \frac{\partial \xi^\lambda}{\partial x'^\nu} \right] + \left[ \frac{\partial \xi^\mu}{\partial x'^\nu} (g_{\lambda \nu} \delta^\lambda_\mu) - \frac{\partial \xi^\mu}{\partial x'^\nu} \theta_{\mu \nu} \right]$$

$$+ \frac{\partial \xi^\mu}{\partial x'^\mu} \theta_{\mu \nu} + \theta(\xi^2)$$

Dropping terms of $O(\xi^2)$

$$= g_{\mu \nu} + \frac{\partial \xi^\mu}{\partial x'^\mu} + \frac{\partial \xi^\mu}{\partial x'^\nu} - \frac{\partial \xi_\lambda}{\partial x'^\mu} \left( \frac{\partial x^\nu}{\partial x'^\mu} + \frac{\partial x^\mu}{\partial x'^\nu} - \frac{\partial g_{\nu \lambda}}{\partial x'^\mu} \right)$$

$$= g_{\mu \nu} + \frac{\partial \xi^\mu}{\partial x'^\mu} + \frac{\partial \xi^\mu}{\partial x'^\nu} - 2 \frac{\partial \xi_\lambda}{\partial x'^\mu} \Gamma^\lambda_{\mu \nu}$$

$$g'_{\mu \nu} = g_{\mu \nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$
A transformation of the coordinates which leaves the form of the metric invariant is called an isometry. For example, in $\mathbb{R}^3$ Euclidean space, in Cartesian coordinates $x, y, z$:

$$ds^2 = dx^2 + dy^2 + dz^2 = g_{ij} dx^i dx^j,$$

where $g_{ij} = \delta_{ij}$.

Translations of the coordinates leave lengths invariant:

$$x' = x + c$$

leave $g_{ij} = \delta_{ij}$.

This corresponds to the translational invariance of the Euclidean geometry, which “looks the same” when translated. Similarly, rotations of the coordinates leave $g_{ij} = \delta_{ij}$, corresponding to the rotational invariance of the Euclidean geometry.

In classical mechanics we are familiar with the consequences of such symmetries: translational invariance $\rightarrow$ constant momentum; rotational invariance $\rightarrow$ constant angular momentum.

The existence of isometries in a spacetime has similar consequences for freely falling trajectories.

An infinitesimal coordinate transformation specified by $\delta x^i(x)$ is an isometry if

$$\frac{D_a \delta x^i + D_b \delta x^i = 0$$

Killing Equation.

A solution to this equation is called a Killing vector field.
Consider a geodesic with tangent vector $U^m = {dx^m \over dt}$, where $t$ is the proper time (for a massive object). Along a geodesic:

\[ \frac{dU^m}{dt} \frac{dV^m}{dt} - U^m \frac{dV^m}{dt} U^m = 0 \]

Consider $C = \xi^m U^m$, where $\xi^m$ is a Killing vector field.

\[
\frac{dC}{dt} = \frac{dC}{dt} = U^m \frac{d\xi^m}{dt} + \xi^m \frac{dU^m}{dt} \\
= U^m U^a D_a \xi^m + \xi^m U^b D_a U^m \\
= (0 \text{ by Killing eqn}) + (0 \text{ by geodesic eqn})
\]

\[ \Rightarrow \frac{d}{dt} \left( \xi^m \frac{dx^m}{dt} \right) = 0 \]

Hence, $\xi^m \frac{dx^m}{dt} = C$ is a constant of the motion.

Suppose a metric is independent of one of the coordinates, say $x^0$, so that $\partial g_{00} / \partial x^0 = 0$.

Consider the vector field $\xi^m = \delta^m_0$.

\[ \xi_a = \partial_a \xi^m = \delta_{a0} \]

\[ D_a \xi^m + D^a \xi_m = 0 = \partial_a g_{00} + \partial_0 g_{0a} - \partial_0 \left( \partial a g_{00} + \partial_0 g_{00} - \partial_0 g_{00} \right) \]

Using $\partial a g_{00} = \delta_{a0}$

\[ D_a \xi^m + D^a \xi_m = 0 = \partial a g_{00} + \partial_0 g_{00} - \partial_0 \left( \partial a g_{00} + \partial_0 g_{00} - \partial_0 g_{00} \right) \]

\[ = 0. \]

Hence, $\xi^m$ is a Killing vector field.
Motion in the Schwarzschild Spacetime

\[ ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]

\[ = -dt^2 \quad \text{(for massive object)} \]

The metric is independent of t and \(\phi\), so we can identify two constants of the motion.

\[ t: \quad \delta^m = \delta^0 \rightarrow \delta^m \frac{dx^m}{dt} = 0 \quad \text{and} \quad \frac{dx^m}{dt} = -\left(1 - \frac{2GM}{r}\right) \frac{dt}{dt} = -\tilde{E} \]

\[ \tilde{E} = \left(1 - \frac{2GM}{r}\right) \frac{dt}{dt} \quad \text{is a constant of the motion} \]

\[ = \text{``Energy/unit mass''} \]

\[ \psi: \quad \delta^m = \delta^\phi \rightarrow \delta^m \frac{dx^m}{dt} = \text{curl} \quad \frac{dx^m}{dt} = r^2 \sin \theta \frac{d\phi}{dt} \quad \Rightarrow \tilde{L} = \frac{r^2 \sin \theta}{\omega} \frac{d\phi}{dt} \]

\[ \tilde{L} = r^2 \sin \theta \frac{d\phi}{dt} \quad \text{is a constant of the motion} \]

\[ = \text{``Angular momentum/unit mass''} \]

Along the trajectory of a test particle,

\[ \left(1 - \frac{2GM}{r}\right)(\frac{dt}{dt})^2 - \left(1 - \frac{2GM}{r}\right)^{-1} (\frac{dr}{dt})^2 - r^2 \left(\frac{d\phi}{dt}\right)^2 - r^2 \sin^2 \theta \left(\frac{d\theta}{dt}\right)^2 = K \]

where \( K = 1 \) for massive object (particle)

\[ K = 0 \quad \text{for massless particle} \]

By the symmetry \( \theta \rightarrow \pi - \theta \), if motion begins in the \( \theta = \pi / 2 \) plane, it will remain in that plane. Choose coordinates so that \( \theta = \pi / 2 \).

\[ \Rightarrow \left(1 - \frac{2GM}{r}\right)(\frac{dt}{dt})^2 - \left(1 - \frac{2GM}{r}\right)^{-1} (\frac{dr}{dt})^2 - r^2 \sin^2 \theta \left(\frac{d\phi}{dt}\right)^2 = K \]
\[ \frac{E^2}{(1-\frac{GM}{r})} - \frac{1}{(1-\frac{GM}{r})} \left( \frac{dE}{dt} \right)^2 - \frac{\mathcal{L}^2}{r^2} = K \]

\[ \frac{1}{2} \left( \frac{dE}{dt} \right)^2 + \frac{1}{2} \left( 1-\frac{2GM}{r} \right) \left( K + \frac{\mathcal{L}^2}{r^2} \right) = \frac{1}{2} E^2 \]

Compare with the Newtonian limit:

\[ K = 1, \text{ ignore } \frac{1}{r^2} \text{ term, } \mathcal{L} = t \]

\[ \frac{1}{2} \left( \frac{dE}{dt} \right)^2 + \left[ \frac{1}{2} - \frac{GM}{r} + \frac{1}{2} \frac{\mathcal{L}^2}{r^2} \right] = \frac{1}{2} E^2 \]

\[ V_{NR}(r) \text{ nonrelativistic effective potential} \]

The values of \( E^2/2, \mathcal{L}^2 \) determines the type of trajectory

\[ V_{NR} \]

\[ \frac{1}{2} E^2 \quad \text{unbound trajectory} \]

\[ \frac{1}{2} \frac{GM}{r} \quad \text{Elliptical orbit} \]

\[ \mathcal{L}^2 = \text{Circular orbit} \]

The \( \frac{1}{r^3} \) which we dropped is a correction from GR.

\[ V(r) = \frac{1}{2} K - \frac{GM}{r} + \frac{\mathcal{L}^2}{2r^2} - \frac{GM}{r^3} \frac{\mathcal{L}^2}{r^3} \]

\[ V(r) \]

\[ \frac{1}{2} K \quad \text{collides if } r=0. \]

\[ \frac{1}{2} \frac{GM}{r} \quad \text{unbound} \]

\[ \frac{1}{2} \frac{GM}{r} \quad \text{bound (but orbit does not close)} \]
$v = 0$:

- Stable circular orbit
- Unstable circular orbit

Collides with $r = 0$. 