(3) (*Adopted from Gottfried*) As we will see in future quantum mechanics courses, the operators that we study in this problem play an essential role in the theory of many-fermion systems. They allow us to describe states that conform to the Pauli Principle.

The Hamiltonian of a system is given by

$$H = \sum_{n,m=1}^{k} a_n^\dagger A_{nm} a_m,$$

where A_{nm} are the elements of a Hermitian matrix with eigenvalues E_1, E_2, $\cdots$, E_k. The a's are operators and satisfy

$$\begin{aligned} \{a_m, a_n\} &= 0 \\ \{a_m^\dagger, a_n^\dagger\} &= 0 \\ \{a_m^\dagger, a_n\} &= \delta_{m,n} \end{aligned} \tag{1}$$

1. Show that it is possible to introduce a new set of operators

   $$\alpha_n = \sum_{m=1}^{k} U_{nm} a_m$$

   that also satisfy the commutation relations in Eq. (1), but in terms of which

   $$H = \sum_{n=1}^{k} E_n \alpha_n^\dagger \alpha_n,$$

   and where the U_{nm} are elements of a unitary matrix.

2. Show that the operators $N_n = \alpha_n^\dagger \alpha_n$ (with $n = 1, 2, \cdots, k$) are compatible.

3. Let ν_n be the eigenvalues of N_n. Show that $\nu_n = 0$ or 1.

4. Let $|\nu_n\rangle$ (i.e., $|0\rangle$ or $|1\rangle$) be an eigenket of N_n. Show that

   $$\begin{aligned} \alpha_n|0\rangle &= 0; \\ \alpha_n^\dagger|1\rangle &= 0; \\ \alpha_n^\dagger|0\rangle &= e^{i\theta}|1\rangle, \end{aligned}$$

   where θ is real.