Phys622 Quantum Mechanics *II*

Homework Assignment 0

1/17, 2008

Due: *2/5 Tuesday* in class.

This is a review of wave mechanics materials. View it as a long (and challenging) practice session to warm up for Quantum Mechanics II. Its second purpose is to serve as training/preparation for the Qual-exam. The problems have been carefully selected and designed with these goals in mind. They are meant to deal only with materials you have seen, but the interface may not be perfect. Please do the best you can. Start the assignment early and tough it out. I believe you will look back in the future and find this process extremely helpful.

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(1) A particle of mass m is confined in a one-dimensional box of length a:

$$V(x) = \begin{cases} \infty, & \text{if } x < 0; \\ 0, & \text{if } 0 \le x \le a; \\ \infty, & \text{if } x > a. \end{cases}$$

At $t = 0$ the particle is in a state with normalized wave function

$$\psi(x, t = 0) = \sqrt{\frac{8}{5a}}\ \sin(\frac{\pi x}{a})\ [1 + \cos(\frac{\pi x}{a})]$$

1. Sketch $\psi(x, 0)$.
2. What are the possible results of an energy measurement at $t = 0$? What is the average energy of the system at $t = 0$?
3. What is the wave function $\psi(x, t)$ at a later time t?
4. What is the average energy of the system at a later time t?
5. At time t, what is the probability to find the particle in the left half of the box (i.e., $0 < x < a/2$)? Argue that your result makes qualitative sense at $t = 0$ based on the sketch in (1).
6. At time t, what is the current density (probability flux) $j(x, t)$ at $x = a/2$? Relate it to your answer in part (5).

(2) To a very good degree, the following confining potential can now be realized in a laboratory in layered materials:

$$V(x) = \begin{cases} V_0\delta(x), & \text{if } |x| < a; \\ \infty, & \text{if } |x| > a. \end{cases}$$

Let m denote the mass of the electron moving in this potential and assume $V_0 > 0$. The parameter a, which gives the thinkness of one layer, is known.

1. Without any calculations, use symmetry to argue that there exists a class of energy eigenstates whose eigenvalues and eigenfunctions do *not* depend V_0. Give the actual eigenvalues and eigenfunctions.

2. Now derive the eigenvalues and eigenfunctions for the other class of eigenstates, the ones that *do* depend on V_0. Give as much detail as you can.

3. As the parameter V_0 varies from 0 to ∞, how would the two classes of eigenvalues compare?

(3) (Sakurai Problem 5.1) A simple harmonic oscillator (in one dimension) described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

is subjected to a perturbation $\Delta V = bx$, where b is a real constant.

1. Calculate the energy change in the ground state to the *lowest nonvanishing order* in b.

2. Solve this problem *exactly* and compare with your results obtained from part (1).

(4) The proton is actually not a point charge, but has a charge distribution with a radius of about 0.8×10^{-13} cm. Show that the electronic s-states are vastly more sensitive to the finite nuclear size than states of higher angular momentum. Estimate the energy shift for the $1s$ and $2s$ levels in hydrogen by assuming the proton's charge to be spread uniformly over a sphere. How big are these effects for a mu-meson bound to a proton? (The mass of a mu-meson is about 208 times that of an electron.)

(5) A quantum ping-pong ball of mass m is above a perfectly *inelastic* table. We will consider vertical motion only. The gravitational potential on the ball is given by the usual form mgz, where z is the distance between the ball and the table top. That is, the Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + mgz.$$

Suppose the ping-pong ball is in its ground state.

1. It can be shown that the ground-state energy has the form

$$E_0 = C\, mg(\frac{m^2 g}{\hbar^2})^{\alpha},$$

where C is a dimensionless constant and α is a parameter. Determine the parameter α using dimensional analysis.

2. Now we use the variational method to estimate C. To select a trial wave function, we note that the boundary condition dictates that the wave function vanish at $z = 0$ and $z = \infty$. So a reasonable form is $ze^{-\lambda z^2/2}$, or $ze^{-\lambda z}$. Using these choices, or an even better one of your own, obtain an upper bound for C.

3. For $m = 1$ gram, obtain a numerical estimate of the ground-state energy of the ping-pong ball.

Note: This same set-up can describe other interesting problems, such as an electron moving on a semi-conductor surface, or the bound state of a heavy quark and its anti-quark, which interact through a linear potential.

(6) (Sakurai Problem 3.15) The wave function of a particle subjected to a spherically symmetric potential $V(r)$ is given by

$$\psi(\mathbf{r}) = (x + y + 3z)f(r),$$

where the particle coordinate is given by $\mathbf{r} = (x, y, z)$ and $f(r)$ is a known function of the radial distance r.

1. Is ψ an eigenfunction of $\mathbf{L}^2$? If so, what is the l-value? If not, what are the possible values of l we may obtain if we measure $\mathbf{L}^2$?

2. What are the probabilities for the particle to be found in various m states?

3. Suppose it is known somehow that $\psi(\mathbf{r})$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.