Chapter 3

Math for Astronomy Review

Astronomy is very much a mathematical science. Astronomers cannot go out 'into the field' like geologists. Nor can they do many 'tabletop experiments' like physicists. Astronomers must rely on reason, logic, and indirect observation. There are only so many pieces of data that one can gather from a galaxy 40 million light-years away that astronomers need to squeeze out every tiny bit of useful information from each observational piece of data. This is why astronomers need to know math. While professional astronomers rely on calculus and higher mathematics, this semester you will not be required to do any math beyond the William and Mary entrance requirement of mathematics. Simply said, if you can properly use a calculator and solve algebraic expressions, you'll be O.K.

Since all of the students here at the College come from varied backgrounds, however, this will serve as a math refresher. The course material will be more interesting for you to learn (and more interesting for us to teach) if we don't get bogged down in the mathematical details and miss the physics. If this review seems familiar and boring, great! If not, then pay close attention, and refer back to this section during the semester. If any of this is completely unfamiliar, your TA can recommend some supplementary materials to get you up to speed.

Topics that will be covered include:

- Algebra
    - ⋆ Order of Operations
    - ⋆ Scientific Notation
    - ⋆ Units and Unit Conversion
    - ⋆ Significant Figures
    - ⋆ Logarithms
- Trigonometry
- Statistics and Feasibility

3.1 Algebra

Equations tell us how different measurements we might make are related. Often we are given a formula in some standard form with n different variables and we only know $(n-1)$ of them. Therefore we must do some algebra and solve for the unknown variable so that we can perform a calculation or make a graph relating two of the variables. Algebra is a requirement for entrance to the College, so we assume you have a basic understanding of the techniques, even if you are a bit rusty. Remember, the basic idea of algebra is that an equation does not change if you perform an operation on both sides of the equals sign. You can add a number to both sides, multiply both sides by the same number, divide both sides by the same number, even exponentiate both sides. If you chose these manipulations strategically, you can isolate one variable on one side of the equation and everything else on the other side.

In the Optics Lab, we're introduced to the *thin lens equation* that governs how lenses focus images. If an object is at a distance of d_o (for $distance_{object}$), a lens of focal length f will produce a clear image at a distance away of d_i, in accordance with:

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} \tag{3.1}$$

Solve this equation for d_i below (show your work):

If we measure d_o to be 1.5 meters and f to be 25 centimeters (0.25 meters), what is d_i?

$$d_i = \underline{\hspace{3cm}}$$

It is in your best interest to do any algebra *before* plugging in any numbers. Otherwise you might make a mistake typing numbers into your calculator (and wouldn't be able to easily go back and find your mistake), or in some cases you might need to perform the same calculation many times, like when you are making a data table. Having the equation solved algebraically will make this much easier. Sometimes rounding errors creep in when algebra could simplify something. Try calculating $\pi \cdot 10$ with your calculator, but only save 6 decimal places. Now divide that number by $\pi \cdot 2$. Compare that number to

$$\frac{10\pi}{2\pi} = \frac{\cancel{\pi} \cdot 10}{\cancel{\pi} \cdot 2} = \frac{10}{2} = 5$$

We get an exact number without any calculator-punching.

Example:

We want to compare the light gathering power of a telescope with diameter of 14" to that of the human pupil ($d \approx 0.20$").

The amount of light that a telescope can gather is dependent on the area of the telescope's primary lens, $A = \pi \cdot r^2$.

$$\frac{L.G.P._{telescope}}{L.G.P._{eye}} = \frac{\pi \cdot r^2_{telescope}}{\pi \cdot r^2_{eye}}$$

Now, you could solve this problem by first finding r, then calculating r^2 and multiplying by π. But you're doing unnecessary work. Notice that the factors of π cancel:

$$= \frac{\cancel{\pi} \cdot r_t^2}{\cancel{\pi} \cdot r_e^2} = \left(\frac{r_t}{r_e}\right)^2 = \left(\frac{2 \cdot r_t}{2 \cdot r_e}\right)^2 = \left(\frac{d_t}{d_e}\right)^2$$

Now we have much less to type into our calculator.

Order of Operations

Everyone should remember Order of Operations. Parentheses, Exponents, Multiplication, Division, Addition, Subtraction. Many of you may remember this with the mnemonic device "Please Excuse My Dear Aunt Sally." When the time comes for you perform some calculations on your calculator, any order-of-operations-mistakes will drastically change your answer. Lets work out some examples, and if you do not arrive at the given answer when you work out the problems on your calculator, then ask your lab partner or T.A. what order of operations you should use.

Examples:

- In error analysis, if the random and systematic errors' values are known (something we'll explore in later labs), then to combine them we add them in quadrature. If the random error is r and the systematic error is s, this means the total error E is given by the formula $E = \sqrt{s^2 + r^2}$.
  - ⋆ If $s = 0.03\ m$ and $r = 0.04\ m$, then what is E? ($E = 0.05\ m$)
- To calculate the distance to a star from its Absolute Magnitude (M) and Apparent Magnitude (m), the formula is $D = 10 \times 10^{(m-M)/5}$ where D is in parsecs.
  - ⋆ If $M = 3.6$ and $m = 4.0$, then what is D? ($D = 12.02pc$)

Scientific Notation

We deal with large ranges of values in the sciences, and to convey large numbers in an understandable way we use Scientific Notation. Lets say we wanted to share with someone that the mass of the moon, to the best of our measuring ability is about 73,600,000,000,000,000,000,000 kg. Say our friend wanted to compare that to the mass of the Earth, which is about 6,000,000,000,000,000,000,000,000 kg. With this many zeroes it is hard to picture how much larger the Earth is than the Moon. If we notice, though, that the mass of the moon is $736 \cdot 100,000,000,000,000,000,000\ kg$, and that the second number is also equal to 10^{20} (just count the number of zeroes and that number is the exponent), then we can more easily write the mass of the moon as $736 \times 10^{20}\ kg$. As a physics and astronomy convention, we only want one number to the left of the decimal place, so notice again that $736 = 7.36 \cdot 100 = 7.36 \times 10^2$. The mass of the moon, in scientific notation, then, is known to be $7.36 \times 10^{22}\ kg$. With the same method, the mass of the Earth is $6.00 \times 10^{24}\ kg$. The beauty of scientific notation comes out when we want to find a ratio of these two masses. The ratio is

$$\frac{Earth\ Mass}{Moon\ Mass} = \frac{6.00 \times 10^{24}\ kg}{7.36 \times 10^{22}\ kg} = \frac{6.00}{7.36} \cdot \frac{10^{24}}{10^{22}} = (0.82) \cdot (10^2) = 8.2 \times 10^1$$

So the Earth is 82 times more massive than the Moon. Metric prefixes work on the basis of powers of ten, so be careful with scientific notation and units. They should flow back and forth in your mind.

$$3.0 \times 10^8\ m/s = 3.0 \times 10^5\ km/s = 3.0 \times 10^{10}\ cm/s$$

Check in Appendix A for a table of metric prefixes and their meanings.

Units and Unit Conversion

Without units, physics is just math. The units are what give equations meaning. If I say that the speed of light is 3, what does that mean? It makes more sense to say that it's 3×10^8 m/s (300 million meters per second). You can treat units just like variables, too. Sometimes they cancel. They can be used to tell you what kind of quantity you're measuring. You should have noticed above that the errors r and s were quoted with units of meters, and that the final answer E also had units of meters. Performing a calculation with correct unit analysis is as important as performing the correct numerical analysis. The good news is that unit analysis follows many of the same rules as numerical analysis. Here are some simple guidelines to keep in mind:

- If you are adding two numbers, they must carry identical units, and their sum will carry the same units
  - $4\ m + 5\ m = 9\ m$
  - $4\ cm + 5\ m = 0.04\ m + 5\ m = 5.04\ m$
- You can multiply and divide units, and the product or quotient will have compound units.

- $4\ N \cdot 5\ m = 20\ N \cdot m$
- $\frac{4\ m}{5\ sec.} = 0.8 \frac{m}{sec.}$

- Multiplying identical units raises the units to a power. Dividing identical units give unit-less quotients.
  - $4\ m \cdot 5\ m = 20\ m^2$
  - $\frac{4\ m}{5\ m} = 0.8$

e.g., if I wanted to know how long it takes light to travel 3,000 m, I just divide:

$$\frac{3 \times 10^3\ m}{3 \times 10^8\ m/s} = 10^{3-8} \cdot \frac{\cancel{m} \cdot s}{\cancel{m}} = 10^{-5} s$$

Remember that when you divide by a fraction, you multiply by the inverse of that fraction:

$$\frac{m}{m/s} = m \cdot \frac{s}{m} = s \cdot \frac{\cancelto{1}{m}}{\cancel{m}} = s$$

Work the following problems with units. Show all of your work.

- The Moon is 395,000 km away from Earth and makes one revolution every 28 days. The speed of an object is the distance it travels divided by the time it takes to travel that distance. What is the speed of the moon, in km/s?

- The mass of an average human is about 70 kg. How many Solar Masses is this? Refer to Appendix A for the mass of the Sun.

Unit Conversion

To convert measurements between unit conventions, in order to complete calculations, it is helpful to multiply by 1. Multiplying by 1 does not change anything, but if you choose your 1 carefully, everything works out. Lets say we wanted to find the mass of a star that is 4 times as large as the sun, but in units of kilograms. We use the fact that

$$1\ solar\ mass = 2 \times 10^{30}\ kg$$

to solve for 1. We call this a *Conversion Factor*:

$$1 = \frac{2 \times 10^{30}\ kg}{1\ solar\ mass}$$

$$4\ solar\ masses \cdot \left(\frac{2 \times 10^{30}\ kg}{1\ solar\ mass}\right) = 4 \cdot (2 \times 10^{30}\ kg) = 8 \times 10^{30}\ kg$$

You can use this technique several times in one calculation. Chemists call this technique 'Stoichiometry'.

As another example, we'll convert the speed of 16 meters/millisecond to the useless (but fun) unit of *furlongs per fortnight*.

1. We first write 16 m/ms as a fraction: $\frac{16\ m}{1\ ms}$
2. Then we multiply by 1 be evoking *conversion factors*. We'll first convert *meters* to *furlongs*. There are 8 furlongs in a mile.
$$\frac{16\ m}{1\ ms} \cdot \frac{1\ mile}{1609\ m} \cdot \frac{8\ furlongs}{1\ mile}$$
3. Notice the units we don't want cancel out:
$$\frac{16\ \cancel{m}}{1\ ms} \cdot \frac{1\ \cancel{mile}}{1609\ \cancel{m}} \cdot \frac{8\ furlongs}{1\ \cancel{mile}} = \frac{7.96 \times 10^{-2}\ furlongs}{ms}$$
4. Now, convert milliseconds to fortnights:
$$\frac{7.96 \times 10^{-2}\ furlongs}{1\ ms} \cdot \frac{1000\ ms}{1\ s} \cdot \frac{60\ s}{1\ min} \cdot \frac{60\ min}{1\ hr} \cdot \frac{24\ hr}{1\ day} \cdot \frac{14\ days}{1\ fortnight}$$
5. Again, notice that our unwanted units cancel and we're left with:
$$\frac{7.96 \times 10^{-2}\ furlongs}{1\ \cancel{ms}} \cdot \frac{1000\ \cancel{ms}}{1\ \cancel{s}} \cdot \frac{60\ \cancel{s}}{1\ \cancel{min}} \cdot \frac{60\ \cancel{min}}{1\ \cancel{hr}} \cdot \frac{24\ \cancel{hr}}{1\ \cancel{day}} \cdot \frac{14\ \cancel{days}}{1\ fortnight}$$
$$= 9.62 \times 10^{7}\ furlongs/fortnight$$

If one of the units is raised to a power, then you must raise its conversion factor to that power, too. *e.g.*, How many cubic nanometers of water can fit in a 1-liter bottle?

Recall from your Chemistry days:

$$1\ l = 1000\ ml = 1\ cm^3 = (1\ cm)^3$$

Then,

$$(1\ \cancel{cm})^3 \cdot \left(\frac{1\ \cancel{m}}{100\ \cancel{cm}}\right)^3 \cdot \left(\frac{10^9\ nm}{1\ \cancel{m}}\right)^3 = \left(\frac{10^9}{100}\right)^3 nm^3 = 10^{21}\ nm^3$$

Some combinations of units are so common that they are given a special name and abbreviated. One example is the Newton. The Newton is a unit of force (abbreviated N) that is shorthand for $1\frac{kg}{m/s^2}$, or $1\ kg\ m^{-1}\ s^{-2}$.

- Perform the following conversions. Refer to Appendix A for any conversion factors you might need.
  - ⋆ 1.0 *lightyear* to *nanometers*
  - ⋆ 15 billion *years* to *seconds*
  - ⋆ 65 *miles per hour* to *meters per second*
  - ⋆ $656.3 nm^{-1}$ to Å^{-1}
  - ⋆ Convert $1.0 \frac{yr^2}{AU^3}$ to $\frac{s^2}{m^3}$

Significant Figures

Let's do a really simple thought experiment. Imagine you're given a spherical ball and you're told to find its radius. You could get out a ruler and try to directly measure the radius, but you're going to be clever. You're going to submerge the ball in water, measure its volume, and work backwards from $V = \frac{4}{3}\pi r^3$ to determine the radius.

You perform this experiment and discover that the ball's volume is $97.7\,cm^3$. Now, determine the radius.

You may have the urge to write down every digit on your calculator's display, but that would be reporting the answer more precisely than we could possibly know.

How thick is the line that marks *1 cm* on a ruler? It's about half a millimeter. How could you measure a distance as $d = 2.8575\ cm$ if your equipment is only precise to $0.5\ mm$? You can't. There must be a logical limit to the accuracy with which you can measure something.

Since our precision is limited by our equipment, we should only say that $d = 2.9\ cm$. Writing an answer with more precision is just wrong.

Here are some simple guidelines to follow to ensure that you do not report more measurement accuracy than you have.

- For 'naked' numbers (numbers which are not a consequence of calculations):
  - ⋆ Leading zeros are not significant (0.000004 has 1 significant figure)
  - ⋆ Trailing zeros are only significant if they are preceded by a decimal point, or are between a digit and a decimal point (405.00 has 5 significant figures)
  - ⋆ Every digit other than zero is significant
  - ⋆ Exact numbers have an infinite amount of significant figures (e.g. We have exactly 1.00000000 moon)
- For numbers that result from mathematical operations:
  - ⋆ The number of significant figures for a product or quotient is equal to the smallest number of significant figures contained in the numbers being multiplied or divided. ($\frac{45.7}{8.27643} = 5.52$, $\frac{1.0}{8.0000} = 0.13$)
  - ⋆ The number of significant figures *to the right of the decimal place* for a sum or difference is equal to the smallest number of significant figures *to the right of the decimal place* contained in the numbers being added or subtracted.($5.2 + 8.2158 = 13.4$)
  - ⋆ For formulas containing more than one operation (most of them will), determine the number of significant figures after each individual operation, and carry that number through to the subsequent operation.

Doesn't that sound easy? It's really not that bad, and if you only remember to keep the same number of significant figures in a final answer as the least precise number within the calculation, usually you'll be O.K.

Logarithms and Powers of 10

For 'common' logarithms, the logarithm of a number is the power to which 10 must be raised to obtain the number:

$$if\ \ y = 10^x, \qquad then\ \ x = log\ y$$

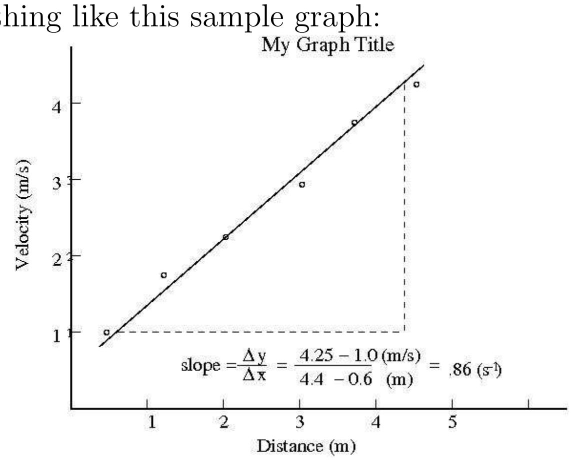
For example log 10^5 = 5.00 and log 2 = 0.3010. Raising a number to a power of 10 is the inverse operation from taking a logarithm so that Log(10^x) = 10^{Logx} = x. Some useful relations using logarithms are:

if N = A·B	log N = log A + log B
if N = A/B	log N = log A - log B
if N = A^n	log N = n log A

Solve the equation $d = 10 \times 10^{(m-M)/5}$ for M and simplify.

Graphs and Plotting

Graphs are often the most important part of your report and an excellent way to display mathematical relationships (functions). There are several things to keep in mind when you make a graph either on the computer or drawn by hand. It should fill most of the page if drawn by hand. Both axis should be labeled with a axis title and units, and have tick marks. Data points should have error bars if appropriate. If you draw a 'best fit' line through the data, it should be your estimate of the line that passes nearest to all the points. It should not necessarily connect the first and last points or be a 'connect the dots' series of line segments. If you take a slope, you should indicate on the graph what points on the line (not single data points) were used for the vertical and horizontal difference as well as the final value. A graph should look something like this sample graph:

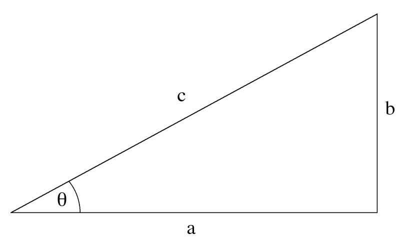


3.2 Trigonometry

You should know the difference between *radians* and *degrees*. We use both in astronomy, but typically for different reasons. When describing observations, it is more instructive to use degrees; for calculations, we use radians. Too many people hate trigonometry because they think that it's too much to memorize. For Astronomy, you really only need to remember a few simple things; everything else trigonometric can be done on your calculator.

θ (degrees)	θ (radians)	$\sin(\theta)$	$\cos(\theta)$
0	0	0	1
90	$\frac{\pi}{2}$	1	0
180	π	0	-1
270	$\frac{3\pi}{2}$	-1	0
360	2π	0	1

Much of trigonometry is based on the right triangle:



The following are relations that you should recall:

$$\begin{aligned} \text{Pythagorean Theorem:} \quad & a^2 + b^2 = c^2 \\ \text{sine:} \quad & \sin(\theta) = b/c \\ \text{cosine:} \quad & \cos(\theta) = a/c \\ \text{tangent:} \quad & \tan(\theta) = b/a \\ \\ \text{arcsine:} \quad & \theta = \arcsin(b/c) \\ \text{arccosine:} \quad & \theta = \arccos(a/c) \\ \text{arctangent:} \quad & \theta = \arctan(b/a) \end{aligned}$$

In Astronomy, we usually deal with really small angles. When the angle θ is small, then $c \approx a$. If we measure θ in radians (denoting this as θ^r), then we can say:

$$\theta^r = \frac{b}{a}$$

This is called the *Small Angle Formula*, and also has another form when we measure θ in *arcseconds*:

$$\theta'' = 206,265 \cdot \frac{b}{a},$$

where the factor of $206,265$ is the number of arcseconds in 1 radian.

- The Moon's angular diameter is about $0.5°$ when viewed from earth. Determine its radius in km. Refer to Appendix A for the distance to the Moon.

3.3 Statistics

No matter how careful we are when we make measurements, we have to deal with uncertainties in our results. These *errors* that we encounter aren't necessarily *mistakes* (although sometimes they are), but are reflections of the limitations of our techniques and equipment.

The science of *Error Analysis* studies how error can appear in and propagate through scientific measurements and calculations. In this class, however, we'll only deal with error analysis at the most basic level. Much of it will probably be a review for you.

Calculating Percent Error

When we make a measurement of some known quantity, we want to know if our result is "good" or not. In order to do this, we compare our answer to the value of the "known" quantity by computing the Percent Error.

Suppose that we take measurements to determine the mass of the Earth and discover that $M = 6.08 \times 10^{24}$ kg. Other scientists have done this; the "accepted" value of M is 5.97×10^{24} kg. We determine our % Error as follows:

$$\begin{aligned}
\%\text{Error} &= \left(\frac{\text{measured} - \text{accepted}}{\text{accepted}}\right) \times 100\% \\
&= \left(\frac{6.08 \times 10^{24} kg - 5.97 \times 10^{24} kg}{5.97 \times 10^{24} kg}\right) \times 100\% \\
&= \left(\frac{6.08 - 5.97}{5.97}\right) \times 100\% \\
&= 1.84\%
\end{aligned}$$

The fact that our % Error is positive indicates that our measured value is higher than the accepted value.

Typically, we take the Absolute Value of % Error.

Dealing with a Data Set: Mean and Standard Deviation

Suppose that you measure a particular quantity a number of times in a repeatable experiment and record the following five numbers:

$$\{3, 5, 3, 4, 5\}$$

Which one is our best estimate for the actual value of the quantity? The answer, as you might expect, is the *arithmetic mean* of all of our number. We define the mean, $\overline{x}$, as the sum of all of the values (x_i) divided by the number of values:

$$\begin{aligned}\overline{x} &= \frac{1}{N} \cdot \sum_{i=1}^{N} x_i \\ &= \frac{3+5+3+4+5}{5} \\ &= 4\end{aligned}$$

How do we determine if our data is "good" or not?

Consider the following two data sets. Both measure the same property:

$$\begin{array}{ll}\text{data set 1:} & \{3,5,3,4,5\} \\ \text{data set 2:} & \{1,7,0,10,2\}\end{array}$$

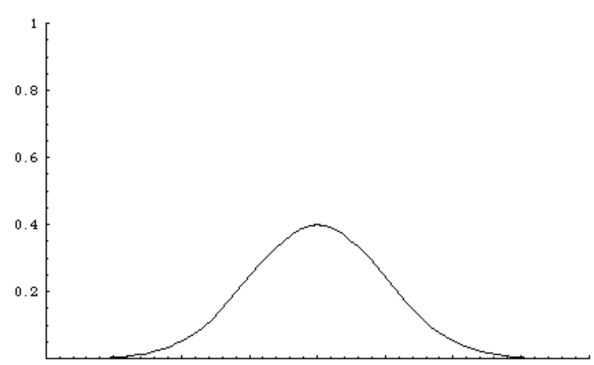
The mean of both sets is 4, but set 2 is *very different* than set 1—it's a lot more spread out. We have a way to quantify this, and it's called *Standard Deviation.*

To calculate the standard deviation of a set of data, you first calculate the mean, $\overline{x}$, and then perform the following procedure:

1. Calculate the *deviation* of each measured value from the mean: $(x_i - \overline{x})$
2. Square the deviation: $(x_i - \overline{x})^2$
3. Add up all of the deviations, divide by the number, and take the square root of this. The result is the *Standard Deviation* (σ):

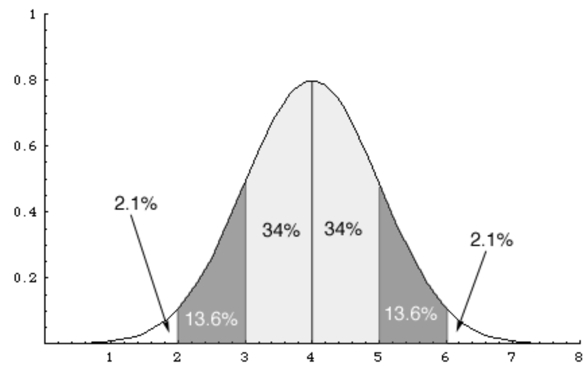
$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2}$$

As we've said, the standard deviation is a measurement of how *spread out* your data is. This is best illustrated by the following example: We'll take a lot of measurements of some physical property and plot their distributions in a *frequency plot*:



Notice that most of the data points cluster around the mean, which is 4. The data in this plot have a standard deviation of 1. There's a rule, called the *Empirical Rule*, that says that 68% of your data will fall within 1 standard deviation of the mean. *i.e.*, 68% of our data fall between 3 and 5. About 95% of the data will fall within 2σ of the mean. and 99.7% of the data will fall within 3σ of the mean.

We depict this in the following graph:



Thus, a set of data with a lower standard deviation is clustered more closely around the mean.

You measure the Sun's rotational period five times and gather the following data set (in days):

$$\{23.3, 27.1, 29.9, 30.3, 23.5\}$$

- What is the mean of this data set?

- Calculate the Standard Deviation of this data set by hand, showing all work.

Feasibility

Every semester the TAs bang their respective heads against the wall when a student tells them that the Earth travels around the sun at a speed of 4.5×10^{18} m/s, or that the age of the universe is 6.6×10^{-15} m. Please do your TA's forehead a favor and check the feasibility of your answer. If your answer does not make any sense, then check your work for algebra/calculator mistakes. If it still doesn't make sense, then ask your TA. Maybe the answer is *supposed* to be surprising in order to illustrate an interesting concept, or maybe you made a fundamentally bad assumption. If you do not pay attention to what your numbers are saying, you might miss out on some big-picture concept that changes your perspective on the universe.

Putting Everything Together

In the Kepler's Laws lab, we will learn that the speed of an object in orbit around another object depends on three parameters: the distance between the objects (r), the mass of the central object (M), and the semi-major axis of the orbit (a):

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}, \tag{3.2}$$

where v is in $\frac{meters}{second}$, M is in *kilograms*, and both r and a are in *meters*.

- Determine the units of G ____________.
- Solve Equation (3.2) for v^2.

- The following data table lists hypothetical values of $v(m/s)$ and $r(m)$ for the Comet Halley. The semi-major axis for Halley's comet is a = 17.8 AU. Fill in the columns

v (m/s)	**v^2 (m^2/s^2)**	**r (m)**	$\frac{2}{r}$ **(m^{-1})**	$\frac{2}{r} - \frac{1}{a}$ **(m^{-1})**
915		5.25×10^{12}		
5101		3.52×10^{12}		
8741		2.10×10^{12}		
18444		6.78×10^{11}		
13220		1.18×10^{12}		
7860		2.34×10^{12}		
4760		3.67×10^{12}		

  for v^2 and $\frac{1}{r}$. Using the graph paper at the end of this lab, construct a plot of v^2 vs. $(\frac{2}{r} - \frac{1}{a})$. Draw a line of best fit and determine the slope of this line, showing any work.

- From your answer to above, you should see that the slope of this line should be equal to GM. If $M = 1.99 \times 10^{30} kg$, compute a value for G. Remember units!

- The accepted value for G is 6.673×10^{-11} in standard units. Compute your % error.