

# Chapter 8

## Rotational Motion

### 8.1 Purpose

In this experiment, rotational motion will be examined. Angular kinematic variables, angular momentum, Newton's 2<sup>nd</sup> law for rotational motion, torque, and moments of inertia will be explored.

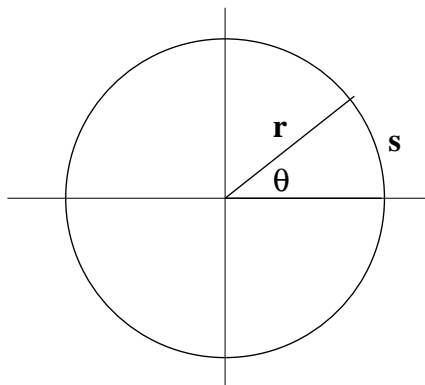
### 8.2 Introduction

**Note: For this experiment, you will write a complete (formal) lab report and hand it in at the next meeting of your lab section. This lab can not be your dropped grade for the semester.**

It is possible to use linear velocity and acceleration to describe angular motion, but it is not convenient, since the linear velocity and acceleration depends on the distance from a rotation axis. In fact, angular quantities can be used to describe rotational kinematics and dynamics in (almost) complete analogy with linear kinematics and dynamics.

The angular displacement of a rigid object rotating about a fixed axis is the angle the object turns,  $\Delta\theta$ . The convention is a positive displacement is counterclockwise and negative if clockwise. The units are radians (rad) which are dimensionless.

A radian is the arc length divided by the radius. Since the total arc length around a complete circle (rotation) is the circumference ( $2\pi r$ ), there are  $2\pi$  radians in  $360^\circ$ . That works out to  $1 \text{ rad} = 57.3^\circ$ .



$$\theta = \frac{s}{r}$$

$$2\pi \text{ (rad)} = 360^\circ$$

The rate of change of the angle with respect to time,  $\Delta\theta/\Delta t$ , is the same for all parts of a revolving body. This quantity is called angular velocity,  $\omega$ . It is vector quantity which is directed along the axis of rotation.

$$\omega = \frac{\Delta\theta}{\Delta t} \quad (8.1)$$

The units of angular velocity are  $\text{s}^{-1}$  (radians/s) since a radian is dimensionless.

The rate of change of angular velocity is called **angular acceleration**  $\alpha$ :

$$\alpha = \frac{\Delta\omega}{\Delta t} \quad (8.2)$$

The units are  $\text{s}^{-2}$  (radians/ $\text{s}^2$ ).

There is a simple connection between linear and angular quantities. The relations of linear variables to angular variables are:

$$s = r\theta$$

$$v = r\omega$$

$$a = r\alpha$$

Note that in using these relationships, radians must be used.

If we have **constant** angular acceleration, everything is mathematically similar to linear kinematics with constant acceleration except position is replaced by angle, velocity is replaced by angular velocity and constant acceleration is replaced by constant angular acceleration. The kinematic important equations for constant angular acceleration,  $\alpha$  are:

$$\omega_f = \omega_0 + \alpha t \quad (8.3)$$

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \quad (8.4)$$

Torque is 'rotational force'. Torque is a vector but our experiment will not require a detailed investigation of torque's vector nature. Torque,  $\tau$ , is defined as:

$$\text{Torque} = (\text{Magnitude of the force}) \times (\text{Lever arm}) \quad (8.5)$$

$$\vec{\tau} = FL \quad (8.6)$$

where the level arm,  $L$ , is defined as the perpendicular distance between the 'line of action' and the axis of rotation. The 'line of action' is an imaginary line collinear with the force. The units of torque are N m. (**Note this is not a joule**). The direction of a torque is positive if it tends to produce a counterclockwise (ccw) rotation. If the torque tends to produce a clockwise rotation, the torque is negative.

The general definition of torque is given by the 'cross product' of the position vector,  $\vec{r}$  and the force,  $\vec{F}$ :

$$\vec{\tau} = \vec{r} \times \vec{F} \quad (8.7)$$

Newton's second law is also true for rotational motion. However, the mass in Newton's second law for linear motion ( $\vec{F} = m\vec{a}$ ) must be replaced by the moment of inertia. Newton's second law for rotational motion is:

$$\Sigma\tau = I\alpha \quad (8.8)$$

where  $\Sigma\tau$  is the sum of all torques acting on the rigid body,  $\alpha$  is the angular acceleration and  $I$  is the moment of inertia.

Normally the mass is the easy part. With rotational motion, the 'mass' part is the moment of inertia. Different shaped extended objects have different formula for the moment of inertia. Note the moment of inertia also depends on which axis the object is rotating about. The units are  $\text{kg m}^2$ .

The general definition of the moment of inertia is obtained from the square of the radius weighted by the object's mass density:

$$\Sigma I = \int \rho(r)r^2 dV \quad (8.9)$$

where  $\rho(r)$  is the mass density,  $r$  is the position, and  $dV$  is the volume element. In general,  $I$  varies as the square of a radius and is proportional to the mass.

For our purposes, we will need the moment of inertia for two geometries. The moment of inertia of a disk (or cylinder) is  $\frac{1}{2}mr^2$  where  $m$  is the total mass and  $r$  is the radius of the disk. The moment of inertia of a point particle is  $mr^2$  where  $m$  is the mass of the particle and  $r$  is the radius of rotation of the particle.

The most general definition of angular momentum,  $\vec{L}$  is:

$$\vec{L} = \vec{r} \times \vec{p} \quad (8.10)$$



Figure 8.1: The rotational motion apparatus. The two steel and one aluminum disk are shown along with the bobbin, barbell and other items

where  $\vec{p}$  is the linear momentum. Angular momentum can also be expressed in analogue with linear momentum as:

$$L = I\omega \quad (8.11)$$

where  $I$  is the moment of inertia and  $\omega$  is the angular velocity.

## 8.3 Procedure

Figure 8.1 shows the components of the rotational motion apparatus. The apparatus will be used to investigate three different aspects of rotational motion.

### 8.3.1 Inelastic Sticking of Two Rotators - Angular Momentum

Two rotating bodies that come together and stick will subsequently rotate with a common final angular velocity. To conserve angular momentum:

$$I_1\omega_{1i} + I_2\omega_{2i} = (I_1 + I_2)\omega_f \quad (8.12)$$

where  $I_1$  ( $I_2$ ) is the moment of the inertia for the first (second) body,  $\omega_{1i}$  ( $\omega_{2i}$ ) is the initial angular velocity of the first (second) body, and  $\omega_f$  is the final angular velocity common to both bodies. If one disk is initially at rest ( $\omega_{2i}=0$ ) then

$$\frac{\omega_{1i}}{\omega_f} = \frac{I_1 + I_2}{I_1} \quad (8.13)$$

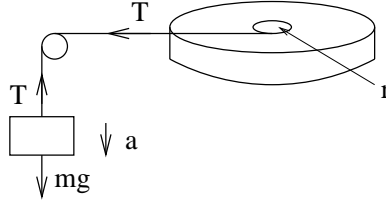



Figure 8.2: This schematic shows the arrangement of the apparatus to determine the angular acceleration from a known torque

- Weigh the two steel disks and the aluminum disk. Measure the radius of each disk. Verify the apparatus is level. Place the steel disk with the large center hole onto the base. The larger end of the hole of this disk should be up. Add the second steel disk (with small hole) on top of the first disk. Open the file 'rot\_motion' in Capstone. A plot of angular velocity (in radians/second) vs time will appear.
- Turn on the air pump. With the air vent drop pin inserted to force air between the two disks, spin by hand the top disk. The bottom disc should remain stationary.
- Click 'record' to begin recording data. After a few seconds, remove the air vent pin. Both discs should move at a common angular velocity .
- Calculate the moment of inertia for both disks using  $\frac{1}{2}mr^2$ . Using the statistic icon () , determine  $\omega_{1i}$  and  $\omega_f$ . You must select the line you want to measure using the legend before highlighting a region. Compare the experimental value of  $\frac{\omega_{1i}}{\omega_f}$  to the value calculated from equation 8.13.
- Repeat with the aluminum disk.
- Save the data as '.cap' file and include the graphs in your lab report.

### 8.3.2 Angular Acceleration from a Torque

A torque,  $\tau$ , arising from a force,  $T$ , applied to a rotor with moment of inertia,  $I$ , produces an angular acceleration,  $\alpha$ :

$$\tau = Tr = I\alpha \quad (8.14)$$

where  $r$  is the perpendicular distance between the applied force,  $T$ , and the rotor's axis. See Figure 8.2.

In figure 8.2, a mass,  $m$ , is suspended by a string wound on a bobbin of radius,  $r$ , attached to the top disk. String tension provides the force,  $T$ , which is always perpendicular to the radius.  $T$  is given in terms of the resulting angular acceleration,  $\alpha$ , by:

$$T = \frac{I\alpha}{r} \quad (8.15)$$

Now consider the linear (vertical) motion of  $m$ . Newton's 2<sup>nd</sup> law gives:

$$T - mg = m(-a) \quad \rightarrow \quad T = mg - ma \quad (8.16)$$

The relationship between angular acceleration and linear acceleration is:

$$a = \alpha r \quad (8.17)$$

Eliminating  $T$  and  $a$  from equation 8.15, equation 8.16 and equation 8.17 results in an expression for the angular acceleration:

$$\alpha = \frac{mgr}{I + mr^2} \quad (8.18)$$

where  $I$  is the moment of inertia of the disk.

- Mount the bobbin wound with a string on top of the steel disk. The radius of the bobbin where the string unwinds is 1.25 cm. Pass the string over the pulley and attach the mass to the end of the string. Wind the string on the bobbin so the mass hangs freely just below the pulley. Keep the mass from falling by holding the top disk with your finger.
- Click 'record' and release the top disk so the mass will fall. Click 'Stop' when the string completely unwinds from the bobbin.
- Fit the data where the mass is falling with a linear fit. The slope of this line is the angular acceleration,  $\alpha$ . Note the units of  $\alpha$  are radians/s<sup>2</sup>.
- Compare the experimentally determined value of  $\alpha$  with the value calculated from equation 8.18.
- Save the '.cap' file and include the plot in your lab report.

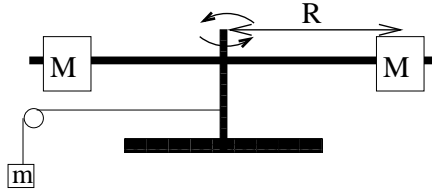


Figure 8.3: This schematic shows the arrangement of the barbell assembly

### 8.3.3 Moment of Inertia - Dependence of I on $R^2$

The moment of inertia for a mass point is  $mR^2$ . In this procedure, the  $R^2$  dependence of the moment of inertia will be investigated.

Consider the situation shown in figure 8.3. The moment of inertia for the entire system is:

$$I_{total} = 2MR^2 + I_0 \quad (8.19)$$

and

$$2MR^2 = I_{total} - I_0 \quad (8.20)$$

where  $I_0$  is the moment of inertia of the disk, bobbin and the barbell **without** the masses,  $M$ .  $2MR^2$  is the moment of inertial of the **two** masses,  $M$ , which rotate around the axis of rotation at a distance  $R$ .

First  $I_0$  will be determined by removing the masses,  $M$ , from the system. Solving equation 8.18 for  $I$  gives:

$$I_0 = \frac{mgr}{\alpha} - mr^2 \quad (8.21)$$

where  $\alpha$  is the angular accelerating of the system,  $m$  is the hanging mass and  $r$  is the radius of the bobbin.

The moment of inertia will then be re-measured with the masses,  $M$ , at three different positions. By subtracting  $I_0$  from these four measures, the value of  $2MR^2$  will be determined.

- Attach the barbell assembly on top of the bobbin. Use the long screw to secure the barbell assembly to the top of the bobbin. The barbell should turn freely. Remove the masses,  $M$ . Weigh the masses.

- Wind the string on the bobbin, click 'record' and allow the mass to fall. Fit the angular speed with a line and determine the angular acceleration,  $\alpha$ . Using equation 8.21, determine the moment of inertia which is  $I_0$  for the system.
- Position the masses at a radius,  $R$ , approximately half the length of their supporting arm. Verify the barbell turns freely with the masses attached. Both masses should be equidistant from the axis of rotation. Before rotating the assembly be sure both masses are clamped tightly to the arm.
- Repeat the measurement of the angular acceleration and determine the moment of inertia from Equation 8.21 for the entire system. Subtract  $I_0$  to determine the moment of inertia of the two masses ( $2MR^2$ ).
- Adjust  $R$  to three other values. Record their values and the resulting values of  $\alpha$  and the moment of inertia of the two masses.
- Plot the measured value of the moment of inertia of the two masses vs  $R$ . You can use excel or the experiment file 'plot.cap'. When using the 'plot.cap' file, fill in the  $R$  values in the x column of the table and the measure moment of inertia of the two masses in the y column. Fit the curve with a quadratic fit. Does the mean error to the fit indicate a good fit?

### 8.3.4 Questions to be Addressed in Your Lab Report

1. Solve equation 8.15, equation 8.16 and equation 8.17 for equation 8.18 and equation 8.21.