Appendix A

Mathematics Appendix

A.1 Units

To measure a physical quantity, you need a standard. Each physical quantity has certain units. A unit is just a standard we use to compare, e.g. a ruler. In this laboratory we will mainly use the SI or metric system. The SI unit of length is the meter (m). The mass unit is the kilogram (kg). The time unit is the second (s). In nearly all laboratory situations, units of meter, kilograms and seconds should be used. From these 'base' units, we will derive all the other quantities we will study, e.g. volume, acceleration, energy, etc. For example, a watt can be expressed as:

\[ 1 \text{ watt} = 1 \text{ W} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^3 \]

Because some of the things we will use are very big or small, i.e. the mass of an atom or the distance to the sun, we often employ scientific notation. This is just expressing the number with powers of ten. For example the distance to the moon is approximately:

\[ \text{distance to moon} = 382,000,000 \text{ m} \]

That is a lot of zeros so we will write it as:

\[ \text{distance to moon} = 3.82 \times 10^8 \text{ m} \]

Proper scientific notation is a number between 1.0 and 9.999... and a power of ten. The number of decimal places we give with a number is called the significant digits. These depend on how well we know the number. In the above distance to the moon, we have 3 significant digits.

In the SI system there are standard prefixes which tell us the power of ten to associate with the unit. For example 'kilo' (k) means \(10^3\), so a kilometer is 1000 meters. Some of the common SI prefixes are:
An important issue is how to convert units. You may be given a time in minutes and want to convert to seconds. The important concept is to multiply the quantity by '1'. For example, 1 m = 3.281 ft so 1 \( \frac{m}{100 \text{ m}} = \frac{1.0 \text{ m}}{3.281 \text{ ft}} \). To convert 57 ft to meters:

\[
57 \text{ ft} = 57 \text{ ft} \times \frac{1 \text{ m}}{3.281 \text{ ft}} = 17.37 \text{ m}
\]

You should always put the units in your calculation and show all conversion factors. If the units come out wrong, e.g. a length in kg, you have done something wrong. In physics, dimension refers to the physical nature of the quantity. Dimensions cancel out just like algebraic quantities, so having the units work out is the best check you can make of your calculation. Some basic conversion factors are given in the following list:

<table>
<thead>
<tr>
<th>Length</th>
<th>1 in = 2.54 cm</th>
<th>Energy</th>
<th>4186 J = 1 kcal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 m = 10^2 cm = 10^3 mm</td>
<td>1 eV = 1.602 \times 10^{-19} J</td>
<td></td>
</tr>
<tr>
<td>Mass</td>
<td>1 kg = 1000 grams</td>
<td>Angle</td>
<td>1 radian = 57.30°</td>
</tr>
<tr>
<td></td>
<td>1 AMU = 1.6605 x 10^{-27} kg</td>
<td>1° = 0.01745 radian</td>
<td></td>
</tr>
</tbody>
</table>

Some important physical constant are given in the following table:

<table>
<thead>
<tr>
<th>Constant</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acceleration of gravity</td>
<td>g</td>
<td>9.80 m/s^2</td>
</tr>
<tr>
<td>Universal gravitational constant</td>
<td>G</td>
<td>6.672 x 10^{-11} N·m^2/kg^2</td>
</tr>
<tr>
<td>Speed of light in vacuum</td>
<td>c</td>
<td>2.997 x 10^8 m/s</td>
</tr>
<tr>
<td>Avogadro’s Number</td>
<td>N_A</td>
<td>6.022 x 10^{23} mol^{-1}</td>
</tr>
<tr>
<td>Electron charge</td>
<td>e</td>
<td>1.602 x 10^{-19} C</td>
</tr>
<tr>
<td>Planck's constant</td>
<td>h</td>
<td>6.626 x 10^{-34} J·s</td>
</tr>
<tr>
<td>Permeability of free space</td>
<td>(\mu_o)</td>
<td>4\pi \times 10^{-7} T·m/A</td>
</tr>
<tr>
<td>Permitivity of free space</td>
<td>(\epsilon_o)</td>
<td>8.854 \times 10^{-12} \frac{C^2}{N·m^2}</td>
</tr>
</tbody>
</table>

### A.2 Graphs and Plotting

Graphs are often the most important part of your report and an excellent way to display mathematical relationships (functions). There are several things to keep in mind when you
make a graph either on the computer or drawn by hand. It should fill most of the page if drawn by hand. Both axes should be labeled with an axis title and units and have tick marks. Data points should have error bars if appropriate. If you draw a 'best fit' line through the data, it should be your estimate of the line that passes nearest to all the points. It should not necessarily connect the first and last points or be a 'connect the dots' series of line segments. If you take a slope, you should indicate on the graph what points on the line (not single data points) were used for the vertical and horizontal difference as well as the final value. A graph should look something like this sample graph:

![Graph Example](image)

### A.3 Statistics

When a measurement is made, it is often useful to compare the measured result to an accepted or standard value. Percent error is used when comparing a result to an accepted value.

\[
\text{\% error} = \left| \frac{X - X_s}{X_s} \right| \times 100\%
\]

where \(X_s\) is the standard or accepted value and \(X\) is the experimental value. Percent difference is used when comparing two results from different experimental methods or measurements. The average of the two measurement is probably closer to the actual value than either measurement. So, the average is used in the denominator.

\[
\text{\% difference} = \left| \frac{X_1 - X_2}{X_{avg}} \right| \times 100\%
\]

where \(X_1\) is one experimental value and \(X_2\) is the other experimental value. The average, \(X_{avg}\) is given by

\[
X_{avg} = \frac{X_1 + X_2}{2}
\]

What if there there are several measured values \(x_1, x_2, \ldots, x_n\)? The **arithmetic mean** is defined as:

\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.
\]
where $\bar{X}$ is the arithmetic mean, $n$ is the number of data points and $X_i$ are the values for the individual points. The mean tells us the average value of all of the measurements. Another important quantity tells us the spread or uncertainty in the set of measurement. First, the deviation is defined by:

$$D_i = X_i - \bar{X}$$

where $D_i$ is the deviation, $X_i$ are the values measured, and $\bar{X}$ is again, the arithmetic mean. There is a Deviation value for each data point and it is either positive or negative in sign. Sometimes the term ‘residual’ is used instead of ‘deviation’. The standard deviation ($\sigma$) is defined by:

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (D_i)^2$$

where $D_i$ is the $i$th deviation and $n$ is the number of measurements. The standard deviation tells us the width or uncertainty in the set of $n$ measurements. The assumption behind this is that any data measurement has an equal likelihood of being higher or lower than the mean. The percent probability of the absolute value of data points falling within integral multiples of $\sigma$ are as follows: 68% - $\sigma$, 95.4% - $2\sigma$, 99.7% - $3\sigma$.

Each individual measure is not exact. The normal method by which uncertainty or error is assigned for single measurements with analog devices such as as voltmeter, meter sticks, etc. is to assume the error in any given measurement is one-half the smallest scale division of the device. Thus, the ruler that is graduated in millimeters has an error of ±0.5 mm. For example, you have measured how far a glider has traveled before it comes to a complete stop. This distance is then determined with a ruler and is found to be 1.234 meters and a small amount more or less than one millimeter. By convention, you are allowed to round up or down to the nearest half millimeter. So, the measurement appears nearer the middle of the distance between the millimeter marks than either side. You will then express the number as 1.2345 meters. If the distance is nearer to the millimeter marks the distance is then 1.234 or 1.235 depending on the position of the glider’s edge. So, to completely state your measured quantity, you can write the following:

$$1.2345 \pm 0.0005 \text{ m}$$

As another example, you have measured the time that the glider was in motion. You found it to be 2.25 seconds by means of a stopwatch that has digits to the hundredth of a second. This is expressed as:

$$2.25 \pm 0.01 \text{ seconds}$$

Note that it is not possible to determine whether or not the time point is nearer the middle of the hundredth or the end marks. The stopwatch only gives whole digits to the hundredth. This is an exception to the $\frac{1}{2}$ smallest scale division rule (ssd) and is applicable to digital measurements made with computers.
When you do calculations with measured numbers to arrive at another number, the error in the calculated number is determined by what calculations are done. If you add two numbers, the error in the sum is different from the error in two numbers you multiply together. The following equations contain formulas for propagating uncertainties through your calculations. Using these basic equations, you can calculate the uncertainty in complicated expressions.

- \( F = x + y \), error: \( (\Delta F)^2 = (\Delta x)^2 + (\Delta y)^2 \).
- \( F = x \cdot y \), error: \( \frac{\Delta F}{F} = \sqrt{(\frac{\Delta x}{x})^2 + (\frac{\Delta y}{y})^2} \).
- \( F = x/y \), error: see expression for \( F = x \cdot y \).
- \( F = x^2 \) error: \( \Delta F = 2x(\Delta x) \).
- \( F = x^n y^m z^k \), error: \( \frac{\Delta F}{F} = \sqrt{(n\frac{\Delta x}{x})^2 + (m\frac{\Delta y}{y})^2 + k(\frac{\Delta z}{z})^2} \).
- \( F = Cx^y \), error: \( \frac{\Delta F}{F} = \sqrt{(y\frac{\Delta x}{x})^2 + (\Delta y \ln x)^2} \).

The above expressions will enable you to express the error equations of all of the calculations you will encounter in this lab. Normally the \( \Delta \) values are the standard deviations (\( \sigma \)) from previous calculations or the ssd rule. These can be substituted back to arrive at expressions for the more complicated equations.

### A.4 Trigonometry

You must be able to do simple algebra and trigonometry. Trigonometry relates the sides and angle for right triangles.

For the angle \( \theta \) the trig functions are defined as:

- \( \sin \theta = \frac{h_o}{h} \)
- \( \cos \theta = \frac{h_a}{h} \)
- \( \tan \theta = \frac{h_o}{h_a} \)
Note that the choice of which side is the 'adjacent' and 'opposite' is determined by the angle.

If you look up the value of a trig function, you can determine the angle. These are called 'inverse' or 'arc' trig functions:

- $\theta = \sin^{-1} \frac{h_o}{h}$
- $\theta = \cos^{-1} \frac{h_a}{h}$
- $\theta = \tan^{-1} \frac{h_o}{h_a}$

Note that $\sin^{-1} \theta$ does not mean $\frac{1}{\sin \theta}$.

Finally, the trig functions are related by the Pythagorean theorem:

\[ h^2 = h_o^2 + h_a^2 \quad \text{or} \quad 1 = \cos^2 \theta + \sin^2 \theta \]