Two-level system

E_a —— $|a\rangle$ or $|1\rangle$ or $|e\rangle$ $|a\rangle, |b\rangle$ - energy eigenstates

Superposition

$$|\psi\rangle = C_a|a\rangle + C_b|b\rangle$$

E_b —— $|b\rangle$ $|2\rangle$ $|g\rangle$

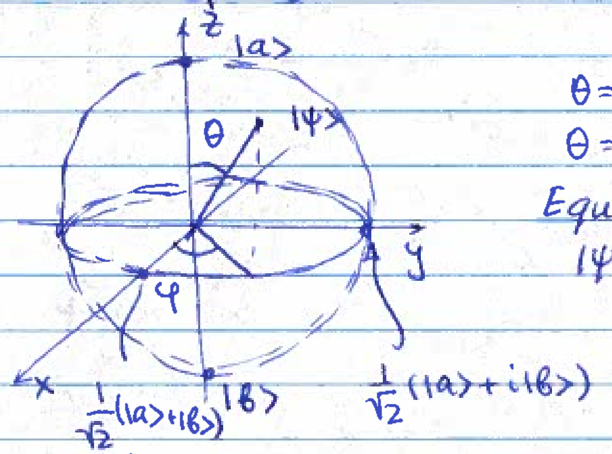
$P_{a,b} = |C_{a,b}|^2$ – probability to find the system in the state $|a\rangle / |b\rangle$

We always assume that the quantum states are normalized $\langle\psi|\psi\rangle = |C_a|^2 + |C_b|^2 = 1$

$$|\psi\rangle = \cos\theta/2\, |a\rangle + e^{i\varphi} \sin\theta/2\, |b\rangle$$

(that guarantees that $|C_a|^2 + |C_b|^2 = 1$)

Visual representation of a two-level system – Bloch sphere



$\theta = 0$ $|\psi\rangle = |a\rangle$

$\theta = \pi$ $|\psi\rangle = |b\rangle$

Equator $\theta = \pi/2$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle + e^{i\varphi}|b\rangle\right)$$

Any quantum superposition corresponds to a point on the surface of the sphere. ~~corresponds to~~

First such visualization was proposed by Poincare to visualize light polarization.

Now it is often used to visualize the dynamics of a qubit or of an electron spin (classical and quantum)

Matrix form

$|a\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|b\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\psi = \begin{pmatrix} c_a \\ c_b \end{pmatrix}$

Interaction Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{\vec{p}} - e\hat{\vec{A}})^2 + \hat{V}_c(\vec{r}) \approx \underbrace{\frac{\hat{\vec{p}}^2}{2m} + V_c(\vec{r})}_{\text{unperturbed atom, } H_0} - \frac{e}{m}\vec{A}\cdot\vec{p} + \underbrace{\cancel{\frac{e^2}{2m}\vec{A}^2}}_{\text{neglect}}$$

Dipole approximation $(r_a \ll \lambda)$ $\frac{e}{m}\vec{A}\cdot\vec{p} \approx \vec{E}\cdot\vec{d}$
(discussed in Scully and Zubairy)

$$\hat{H} = \hat{H}_0 + \hat{H}_I = \hat{H}_0 - \vec{d}\cdot\vec{E} = \hat{H}_0 = - (-e)\vec{r}\,\vec{E}$$

As we discussed last time: we assume EM field to be a weak perturbation

$\hat{H}_0$ describes atomic states
$\hat{H}_I$ describes transitions b/w the states induced by EM field. (and/or shifts of the levels)

Relevant matrix elements

$\langle a|\hat{H}_I|a\rangle = \langle a|+e\vec{r}\vec{E}|a\rangle = 0$, $\langle b|\hat{H}_I|b\rangle = 0$

due to the parity

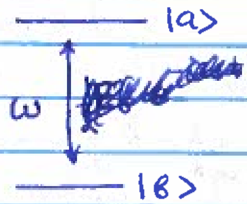
$$\langle a|\hat{H}_I|b\rangle = -\underbrace{\langle a|-e\vec{r}\,\vec{e}_p|b\rangle}_{\text{atom dependent}}\underbrace{E_0 e^{ikz-i\omega t}}_{\text{external}}$$

where $\vec{E} = E_0\vec{e}_p e^{ikz-i\omega t}$ $\vec{e}_p$ – polarization direction

$\langle a|-e\vec{r}\,\vec{e}_p|b\rangle = \wp_{ab}$
$\langle b|-e\vec{r}\,\vec{e}_p|a\rangle = \wp_{ba} = \wp_{ab}^*$

Only polarization direction of e-m field play a role in $\wp_{ab}$ value, the rest depends on atomic state properties.

EM wave interaction with a two-level system



$$E_a = \hbar\omega_a \;,\; E_b = \hbar\omega_b$$
$$\omega_{ab} = \omega_a - \omega_b$$

$$\hat{H} = \hat{H}_0 + \hat{H}_I$$

$$\hat{H}_0 = \begin{pmatrix} \hbar\omega_a & 0 \\ 0 & \hbar\omega_b \end{pmatrix} = \begin{pmatrix} \hbar\omega_{ab} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if } E_b = 0$$

$$\hat{H}_I = -\vec{d}\vec{E} = \begin{pmatrix} 0 & -\wp_{ab}E \\ -\wp_{ba}E & 0 \end{pmatrix} \qquad \mathcal{E} = \frac{1}{2}\mathcal{E}_0 e^{-i\omega t} + \frac{1}{2}\mathcal{E}_0^* e^{i\omega t}$$

$$|\psi\rangle = \begin{pmatrix} c_a \\ c_b \end{pmatrix}$$

$$i\hbar \frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle = \begin{pmatrix} \hbar\omega_{ab} & -\wp_{ab}E \\ -\wp_{ba}E & 0 \end{pmatrix}\begin{pmatrix} c_a \\ c_b \end{pmatrix}$$

$$\begin{cases} i\hbar \dfrac{\partial \dot{c}_a}{\partial t} = \hbar\omega_{ab} c_a - \frac{1}{2}\wp_{ab}\left(\mathcal{E}_0 e^{-i\omega t} + \mathcal{E}_0^* e^{i\omega t}\right) c_b \\ i\hbar \dfrac{\partial \dot{c}_b}{\partial t} = -\frac{1}{2}\wp_{ba}\left(\mathcal{E}_0 e^{-i\omega t} + \mathcal{E}_0^* e^{i\omega t}\right) c_a \end{cases}$$

If no electric field $c_b = \text{constant}$
$c_a \propto e^{i\omega_{ab} t}$

From the classical case we may remember that the induced dipole will oscillate at a frequency of EM field $c_a = \tilde{c}_a e^{-i\omega t}$

Slowly varying amplitude $\tilde{c}_a = c_a e^{i\omega t}$

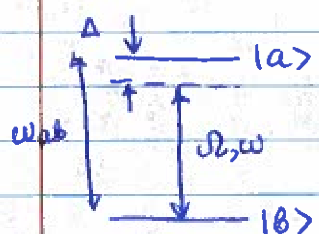
$$\begin{cases} i\hbar\dot{\tilde{c}}_a + i\hbar\omega\tilde{c}_a = \hbar\omega_{ab} c_a - \frac{1}{2}\wp_{ab}\left(\mathcal{E}_0 + \mathcal{E}_0^* e^{2i\omega t}\right) c_b \\ i\hbar\dot{c}_b = -\frac{1}{2}\wp_{ab}\left(\mathcal{E}_0 e^{-2i\omega t} + \mathcal{E}_0^* \cancel{e}\right)\tilde{c}_a \end{cases}$$

Rotating wave approximation: we neglect fast oscillating terms $\sim e^{\pm 2i\omega t}$

$$\begin{cases} \dot{\tilde{C}}_a = i(\omega-\omega_{ab})\tilde{C}_a + i\dfrac{\wp_{ab}\mathcal{E}_0}{2\hbar}c_b \\ \dot{c}_b = i\dfrac{\wp_{ba}\mathcal{E}_0^*}{2\hbar}\tilde{C}_a \end{cases}$$

Common notation $\omega-\omega_{ab}=\Delta$ ~~det~~ frequency detuning of EM field from the transition frequency

$$\Omega = \frac{\wp_{12}\mathcal{E}_0}{2\hbar} \quad \text{Rabi frequency}$$



Since we are going to stay within the rotating wave approximation (RWA), we are going to drop tilde on $\tilde{C}_a \to C_a$, and just remember we are interested in slowly-varying amplitude

$$\begin{cases} \dot{C}_a = i\Delta C_a + i\Omega C_b \\ \dot{C}_b = i\Omega^* C_a \end{cases}$$

On-resonant case $\omega=\omega_{ab}$, $\Delta=0$

$$\begin{cases} \dot{C}_a = i\Omega C_b \\ \dot{C}_b = i\Omega C_a \end{cases} \Rightarrow \ddot{C}_{a,b} = -|\Omega|^2 C_{a,b}$$

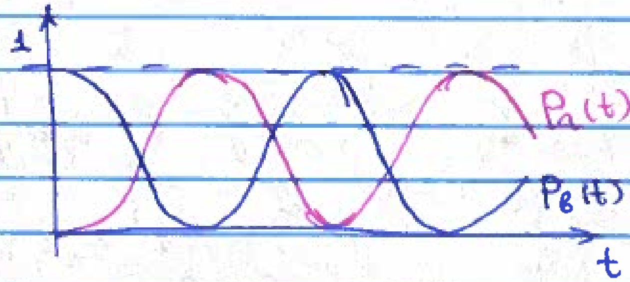
Let's assume, our atom was in the ground state $|b\rangle$ at $t=0$, when EM was turned on

$|C_b|=1 \quad |C_a|=0$

$$C_b(t) = \cos|\Omega|t \qquad C_a = i\frac{\Omega}{|\Omega|}\sin|\Omega|t$$

Populations of each state / probability to find the atom in each of the states is

$P_b = \cos^2 |\Omega| t \qquad P_a = \sin^2 |\Omega| t$

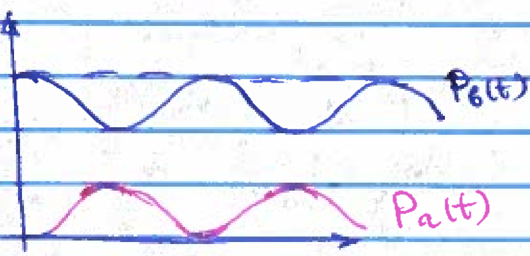


Population oscillates b/w the two states at the frequency Ω

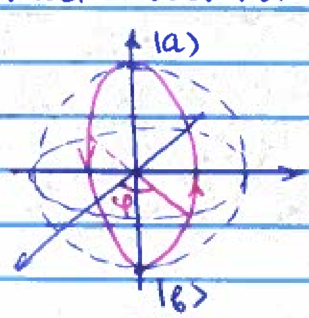
This is the way to describe stimulated absorption and emission.

For $\Delta \neq 0$ – similar oscillatory behavior, but the population transfer is not complete

$$P_a(t) = \frac{|\Omega|^2}{|\Omega|^2 + \frac{1}{4}\Delta^2} \sin^2\left(\sqrt{|\Omega|^2 + 4\Delta^2}\; t\right)$$



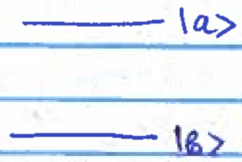
Rabi oscillations on a Bloch sphere



$$|\psi\rangle = \cos|\Omega|t\,|b\rangle + i\overset{e^{i\varphi}}{\frac{\Omega}{|\Omega|}} \sin|\Omega|t\;|a\rangle$$

The atom is "rotated" around one of the meridians of the Bloch sphere

Dressed state formalism



We are operating in RWA

$\omega_{ab} \to \omega_{ab} - \omega = -\Delta$

$$\hat{H} = \begin{pmatrix} -\hbar\Delta & -\hbar\Omega \\ -\hbar\Omega^* & 0 \end{pmatrix}$$

Let's find the eigenstates of $\hat{H}$

$\hat{H}|\pm\rangle = E_\pm|\pm\rangle$

In general $E_\pm = \hbar\left(-\frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + |\Omega|^2}\right)$

1. Resonant case: $\Delta = 0$

$E_\pm = \pm\hbar|\Omega|$ $\quad |\pm\rangle = \frac{1}{\sqrt{2}}\left(|b\rangle \pm \frac{\Omega}{|\Omega|}|a\rangle\right)$

If Ω is real

$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|b\rangle \pm |a\rangle\right)$

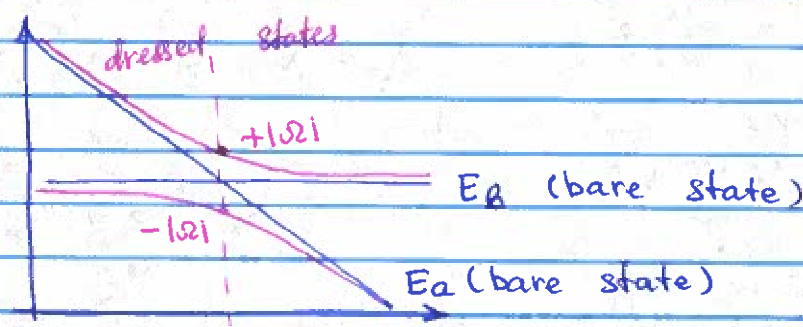
2. Far-detuned case: $\Delta \gg \Omega$

$\Delta > 0$

$E_- \approx -\hbar\Delta - \frac{\hbar|\Omega|^2}{\Delta}$ $\qquad E_+ \approx \hbar\frac{|\Omega|^2}{\Delta}$

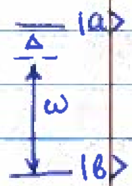
$|-\rangle \approx |a\rangle + \frac{\Omega^*}{\Delta}|b\rangle$ $\qquad |+\rangle \approx |b\rangle - \frac{\Omega}{\Delta}|a\rangle$

Reverses for $\Delta < 0$



The energy splitting b/w the two dressed states on resonance can be used to characterize the strength of the EM wave interaction.

Far-detuned case: adiabatic elimination



$\omega \gg \Delta \gg \Omega$

We expect only a weak effect of EM field on the ground state

$c_b^{(0)} \approx 1$, $|c_a| \ll 1$ (simplified estimate)

$$\begin{cases} \dot{c}_a = i\Delta c_a + i\Omega \overset{\approx 1}{c_b} \approx i\Delta c_a + i\Omega \\ \dot{c}_b = i\Omega^* c_a \end{cases}$$

$c_a = -\frac{\Omega}{\Delta}(1 - e^{i\Delta t})$ rapid oscillations at the frequency Δ

Suppose we are interested in slow dynamics at the scale $t \gg 1/\Delta$. Then we can neglect the rapidly oscillating terms, since there effect averages to zero

$$\langle \dot{c}_a \rangle_{t \gg 1/\Delta} = i\Delta \langle c_a \rangle + i\Omega \langle c_b \rangle$$

The idea of adiabatic elimination is that we can neglect the fast dynamics of the excited state, and only take it into account as far as its effect on the ground state

$\langle \dot{c}_a \rangle \approx 0$ $\quad c_a \approx -\frac{\Omega}{\Delta} c_b$ (i.e. excited state followes the ground state, sometimes this is called adiabatic following)

$\dot{c}_b = i\Omega^* c_a = -i\frac{|\Omega|^2}{\Delta} c_b$

$c_b = c_b(0)\, e^{-i\frac{|\Omega|^2}{\Delta} t}$ looks like an energy shift

ac-Stark shift or light shift $\quad \Delta\omega_{ac} = +\frac{|\Omega|^2}{\Delta}$