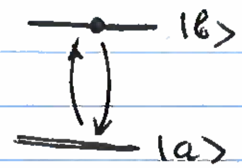
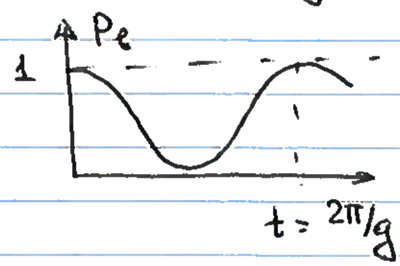
Spontaneous emission in free space

Up to now we restricted optical field to be in one specific mode (which in reality means an atom is strongly coupled to a high-finesse optical cavity)



$$|\psi(t)\rangle = \cos(g\sqrt{n+1}\,t)|b\rangle - i\sin(g\sqrt{n+1}\,t)|a\rangle$$



vacuum Rabi oscillations

This is not a realistic scenario for ~~to~~ vast majority of the situation. Normally, emitted spontaneous photon is gone and is never reabsorbed

More realistic scenario for an atom in a free space is to be coupled to infinite number of vacuum optical modes

$$\hat{\vec{E}}(\vec{k},\omega_k) = \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\; e^{i\vec{k}\vec{r}}\,(\hat{a}_k + \hat{a}_k^{+})$$

Interaction Hamiltonian

$$\hat{H} = \hbar\sum_k \omega_k \hat{a}_k^{+}\hat{a}_k + \frac{1}{2}\hbar\omega_0\,\hat{\sigma}_z + \underbrace{\hbar\sum_k(g_k\hat{a}_k\hat{\sigma}_+ + h.c.)}_{\text{interaction Hamiltonian}}$$

Quantum states of the system

$|b, 0\rangle$ $\qquad\qquad$ $|a, 1_{\vec{k}}\rangle$

$|b\rangle$ $\qquad\qquad$ $\vec{k}, \omega_{\vec{k}}$

no photon

$|a\rangle$ $\qquad\qquad$ $|a\rangle$

$$|\psi(t)\rangle = c_b(t)\,|b, 0\rangle + \sum_{\vec{k}} c_{a,\vec{k}}(t)\,|a, 1_{\vec{k}}\rangle$$

$$c_b(0) = 1, \quad c_{a,\vec{k}}(0) = 0$$

$$i\hbar \frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle \Rightarrow$$

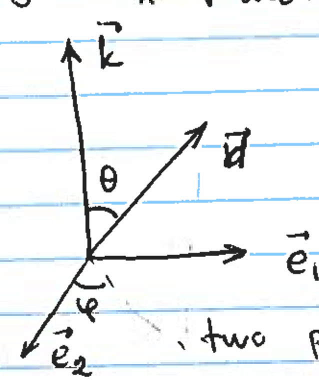
$$\begin{cases} \dot{c}_b(t) = -i \sum_{\vec{k}} g_{\vec{k}}^{*}\, e^{i(\omega_0 - \omega_k)t}\, c_{a,\vec{k}}(t) \\ \dot{c}_{a,\vec{k}}(t) = -i\, g_{\vec{k}}\, e^{-i(\omega_0 - \omega_{\vec{k}})t}\, c_b(t) \end{cases}$$

$$c_{a,\vec{k}}(t) = -i g_{\vec{k}} \int_0^t e^{-i(\omega_0 - \omega_k)t'}\, c_b(t')\,dt'$$

$$\dot{c}_b(t) = -\sum_{\vec{k}} |g_k|^2 \int_0^t e^{i(\omega_0 - \omega_{\vec{k}})(t-t')}\, c_b(t')\,dt'$$

$$\sum_{\vec{k}} \rightarrow \sum_{\vec{e}_{\vec{k}}} \frac{V}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, \underbrace{\int_0^{\infty} k^2\, dk}\ \xrightarrow[k = \frac{\omega_k}{c}]{} \frac{1}{c^3} \int_0^{\infty} \omega_{\vec{k}}^2\, d\omega_k$$

$$g_k = \frac{1}{\hbar} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\ \langle a|\, -\vec{d}\,\vec{e}_{\vec{k}}\, |b\rangle$$



$$\vec{d}\,\vec{e}_1 = |\vec{d}|\sin\theta\cos\varphi$$
$$\vec{d}\,\vec{e}_2 = |\vec{d}|\sin\theta\sin\varphi$$

$$\langle a|\, -|\vec{d}|\, |b\rangle = \wp_{ab}$$

two possible polarizations

$$\sum_{\vec{e}_k} |\vec{g}_k|^2 = \left(\frac{\omega_k}{2\hbar\varepsilon_0 V}\right) \wp_{ab}^2 \quad \sin^2\theta \; \overbrace{\cancel{(\cos^2\varphi + \sin^2\varphi)}}^{=1}$$

no φ dependence

$$\dot{c}_b = -\frac{\cancel{V}}{(2\pi)^{3}\cancel{}2} \underbrace{\int_0^{2\pi} d\varphi}_{\cancel{2\pi}} \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{4/3} \frac{1}{c^3} \int_0^{\infty} \frac{\omega^3 \wp_{ab}^2}{2\hbar\varepsilon_0 \cancel{V}} \int_0^{t} e^{i(\omega_0-\omega)(t-t')} c_b(t')dt'$$

$$\dot{c}_b = -\frac{1}{12\pi^2} \frac{\wp_{ab}^2}{\hbar c^3 \varepsilon_0} \int_0^{\infty} \omega^3 d\omega \int_0^{t} e^{i(\omega_0-\omega)(t-t')} c_b(t')dt'$$

$$\int_0^{\infty} \omega^3 e^{i(\omega_0-\omega)(t-t')} d\omega \approx \omega_0^3 \int_{0\to-\infty}^{\infty} e^{i(\omega_0-\omega)(t-t')} d\omega =$$

$$= \omega_0^3 \cdot 2\pi \, \delta(t-t')$$ Markov approximation

no system memory

$$\dot{c}_b = -\frac{1}{6\pi} \frac{\wp_{ab}^2 \omega_0^3}{\hbar c^3 \varepsilon_0} c_b = -\frac{\Gamma}{2} c_b$$

$$\Gamma = \frac{\wp_{ab}^2 \omega_0^3}{3\pi\varepsilon_0 \hbar c^3} = \frac{1}{4\pi\varepsilon_0} \frac{4\wp_{ab}^2 \omega_0^3}{3\hbar c^3}$$

$$\dot{c}_b = -\Gamma/2 \, c_b \qquad P_b(t) = |c_b(t)|^2 = e^{-\Gamma t}$$

$$c_b = e^{-\Gamma/2 \, t}$$

Irreversible spontaneous decay of the excited states

What about emitted light?

$$c_{ak} = -i g_k \int_0^t dt' \, e^{-i(\omega_0-\omega_k)t'} e^{-\Gamma t'/2} =$$

$$= g_{\vec{k}} \left[\frac{1 - e^{-i(\omega_0-\omega_k)t - \Gamma t/2}}{(\omega_k - \omega_0) + i\Gamma/2} \right]$$

For $t \gg 1/\Gamma$ we can be fairly sure the atom is in the ground state

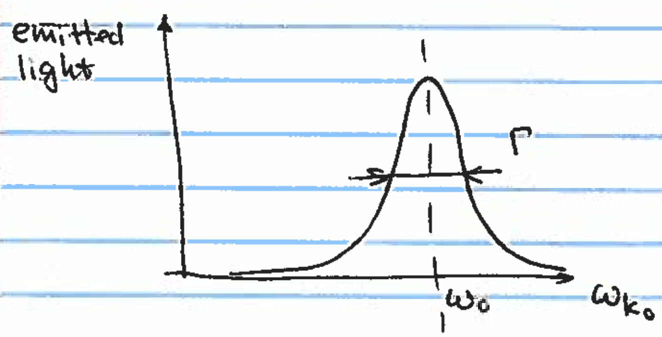
$$|\psi(t \gg 1/\Gamma)\rangle \simeq |a\rangle \sum_{\vec{k}} g_{\vec{k}} \frac{1}{(\omega_k-\omega_0)+i\Gamma/2} |1_{\vec{k}}\rangle$$

Can separate the state of the photon field

$$|sp\rangle = \sum_{\vec{k}} g_{\vec{k}} \frac{1}{(\omega_k-\omega_0)+i\Gamma/2} |1_{\vec{k}}\rangle$$

Spontaneous emission spectral density

$$\langle sp| \hat{E}^{(+)}_{\vec{k}_0} \hat{E}^{(+)}_{\vec{k}_0} |sp\rangle \propto \langle sp| \hat{a}^{\dagger}_{\vec{k}_0} \hat{a}_{\vec{k}_0} |sp\rangle = \frac{|g_{\vec{k}}|^2}{(\omega_{k_0}-\omega_0)^2+(\Gamma/2)^2}$$



Supperradiance and Subradiance

—— $|b\rangle$ —— $|b\rangle$

—— $|a\rangle$ —— $|a\rangle$

atom 1 atom 2

Two indistinguishable atoms, prepared in the entangled state

$$|\psi\rangle_{atom}(0) = \frac{1}{\sqrt{2}}\left(|b,a\rangle \pm |a,b\rangle\right)$$

$$\hat{H} = \hbar \sum_k \cancel{a_k}\, \omega_k \hat{a}_k^{\dagger} \hat{a}_k + \sum_{m=1}^{2}\left[\frac{1}{2}\hbar\omega_0 \hat{\sigma}_z^{(m)} + \hbar \sum_k g_k \hat{a}_k \hat{\sigma}_+^{(m)} + h.c.\right]$$

$$|\psi(t)\rangle = c_{ba0}(t)\,|b,a,0\rangle + c_{ab0}(t)\,|a,b,0\rangle + \sum_k c_{bb1_k}\,|b,b,1_k\rangle$$

Following the same steps, we would get the following coupled equations

$$\begin{cases} \dot{c}_{ba0} = -\frac{\Gamma}{2}\left(c_{ba0} + c_{ab0}\right) \\ \dot{c}_{ab0} = -\frac{\Gamma}{2}\left(c_{ab0} + c_{ba0}\right) \end{cases}$$

$$\frac{d}{dt}\left(c_{ba0} + c_{ab0}\right) = -\Gamma\left(c_{ba0} + c_{ab0}\right)$$

$$\frac{d}{dt}\left(c_{ba0} - c_{ab0}\right) = 0$$

If the initial state of the system

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|b,a\rangle + |a,b\rangle\right) \Rightarrow c_{ab0}(0) = c_{ba0}(0) = 1/\sqrt{2}$

Then the two coefficients will remain equal

and $\frac{d}{dt} c_{ba0} = \cancel{\frac{d}{dt} c_{ba}} - \Gamma c_{ba0} \Rightarrow c_{ba0}(t) = e^{-\Gamma t}$

same for $c_{ab0}(t) = e^{-\Gamma t}$

The entangled atoms decay <u>twice</u> faster than a single atom

Superradiance

If initially the atoms are in the asymmetric state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|b,a,0\rangle - |a,b,0\rangle\right)$

$$C_{ba0} = -\,C_{ab0} = 1/\sqrt{2}$$

then $$\frac{dC_{ba0}}{dt} = \cancel{\frac{d}{dt}}\,\frac{dC_{ab0}}{dt} = 0$$

atoms don't decay at all!
Subradiance