# Classical description and characterisation of fluctuations

## How to define coherence?

A process is coherent if it is characterized by the existance of some well-defined deterministic phase relationship or, in other words, if some phase is not subject of random noise

**Coherent light source (laser)**

$E(z,t) = E_0 e^{ikz - i\omega t + i\varphi(t)}$

The radiation is completely defined by its amplitude and phase.

**Incoherent (thermal, random) light**

Fluorescent light – many independent light sources

$E(t) = E_1(t) + E_2(t) + \ldots = \sum_{i=1}^{N} E_i e^{-i\omega_i t - i\varphi_i(t)}$

(if all sources are identical)

$E(t) = E_0 e^{-i\omega t} \sum_{i=1}^{N} e^{-i\varphi_i(t)}$

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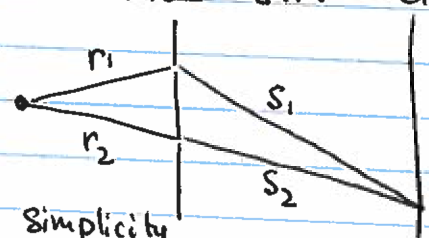
How to characterize degree of coherence of the given source?

Typically we use interference

Double-slit experiment




for simplicity $r_1 = r_2 = 0$

$E_{total}(t) = E_{source}(t - t_1) + E_{source}(t - t_2)$

$t_{1,2} = S_{1,2}/c$

$\tau = |t_1 - t_2| = |S_1 - S_2|/c$

delay b/w the two paths

$$|E_{total}|^2 = |E_s(t-t_1)|^2 + |E_s(t-t_2)|^2 + 2\,\mathrm{Re}\,(E_1^* E_2)$$

In reality, we usually cannot measure instantaneous values of e-m field amplitude. Thus, we need to use a statistical approach, repeating the measurements many times and averaging the results

$$\langle |E_{total}|^2 \rangle = \langle |E_s(t-t_1)|^2 \rangle + \langle |E_s(t-t_2)|^2 \rangle + 2\,\mathrm{Re}\,\langle E_s^*(t-t_1) E_s(t-t_2) \rangle$$

$$\langle |E_s(t-t_1)|^2 \rangle = \langle |E_s(t-t_2)|^2 \rangle = \langle |E_0|^2 \rangle \quad \text{constant}$$

$$E_s(t) = E_0\, e^{-i\omega t - i\varphi(t)}$$

$$E_s^*(t-t_1) E_s(t-t_2) = |E_0|^2\, e^{i\omega\tau}\, e^{i(\varphi(t_1)-\varphi(t_2))}$$

$$\langle \mathrm{Re}\,(E_s^*(t-t_1) E_s(t-t_2)) \rangle = |E_0|^2 \langle \cos(\omega\tau + \varphi(t_1) - \varphi(t_2)) \rangle$$

Ideal laser light $\varphi(t) = \text{const}$ $\quad \tau = \dfrac{|s_1 - s_2|}{c}$

$I \propto \langle |E|^2 \rangle$

$I_{tot} = 2I_s + 2I_s \cos\left\{\frac{\omega}{c}(s_1 - s_2)\right\}$ normal interference

Visibility $V = \dfrac{I_{max} - I_{min}}{I_{max} + I_{min}}$

In the ideal case of coherent light $V = 1$ as $I_{min} = 0$

For describing the interference in general, it is convenient to use first-order correlation function

$$G^{(1)}(t_1, t_2) = \langle E^-(t_1) E^+(t_2) \rangle$$

$$\langle |E_{tot}|^2 \rangle = G^{(1)}(t_1, t_1) + G^{(1)}(t_2, t_2) + 2\,|G_1^{(1)}(t_1, t_2)| \cos(\omega\tau + \varphi(t_1) - \varphi(t_2))$$

$$I_{\substack{max \\ min}} = G^{(1)}(t_1, t_1) + G^{(1)}(t_2, t_2) \pm 2\,|G_*^{(1)}(t_1, t_2)|$$

$$V = \frac{2\,|G^{(1)}(t_1, t_2)|}{G^{(1)}(t_1, t_1) + G^{(1)}(t_2, t_2)}$$

In majority of steady-state sources $\langle |E_s(t_1)|^2 \rangle = \langle |E_s(t_2)|^2 \rangle$ and $|G^{(1)}(t_1, t_2)| = |G^{(1)}(t_2, t_1)|$

Normalized first-order ~~([illegible])~~ coherence function

$$g^{(1)}(\tau) = \frac{\langle E^*(0)\, E(\tau) \rangle}{\langle |E|^2 \rangle}$$

If the source parameter change in time

$$g^{(1)}(t, \tau) = \frac{\langle E^*(t)\, E(t+\tau) \rangle}{\langle E^*(t)\, E(t) \rangle}$$

$$V = |g^{(1)}(\tau)|$$

In ideal case $g^{(1)}(\tau) = 1$

In general $0 \leq |g^{(1)}(\tau)| \leq 1$

Can we see interference from a random source?

$$E(t) = E_0 e^{-i\omega t} \{ e^{i\varphi_1(t)} + e^{i\varphi_2(t)} + \dots e^{i\varphi_N(t)} \}$$

$$\langle E^*(t) E(t+\tau)\rangle = |E_0|^2 e^{-i\omega\tau} \langle (e^{-i\varphi_1} + e^{-i\varphi_2} + \dots e^{-i\varphi_N})( e^{i\varphi_1(t+\tau)} + e^{i\varphi_2(t+\tau)} + \dots$$

By taking an ensemble-average, we will be able to only see oscillations from the same oscillator

$$\langle E^*(t) E(t+\tau)\rangle = |E_0|^2 e^{-i\omega\tau} \sum_{i=1}^{N} \underbrace{\langle e^{i\varphi_i(t+\tau) - i\varphi_i(t)} \rangle}_{\text{single-oscillator correlations}}$$

We will assume that the probability of the oscillator to maintain its phase for time $\tau$ b/w $t$ and $t+\tau$ is $p(\tau)\, d\tau = \frac{1}{\tau_0} e^{-\tau/\tau_0} d\tau$

$\tau_0$ - average coherence time

$$\langle e^{i\varphi_i(t-\tau) - i\varphi_i(t)} \rangle \Rightarrow \begin{cases} 1 & \text{phase didn't jump} \\ 0 & \text{phase did jumped} \end{cases}$$

$$\| \qquad\qquad$$

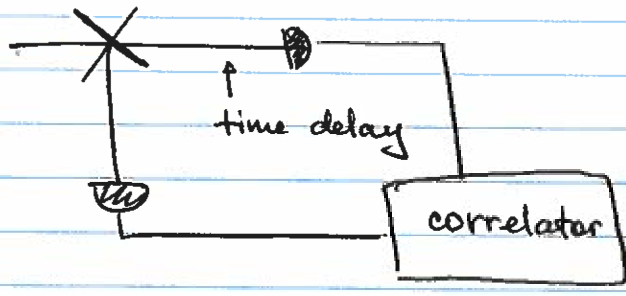
$$\int_\tau^\infty p(\tau')\, d\tau' = e^{-\tau/\tau_0}$$ - probability that the phase is maintained beyond $\tau$

$$\langle E^*(t) E(t+\tau)\rangle = N|E_0|^2 e^{-i\omega t} e^{-\tau/\tau_0}$$

$$|g^{(1)}(\tau)| = e^{-\tau/\tau_0}$$

Second-order coherence

Hanbury-Brown-Twiss experiment




$$G^{(2)}(\tau) = \langle I(t) I(t+\tau)\rangle = \langle E^*(t) E^*(t+\tau) E(t+\tau) E(t)\rangle$$

Normalized second-order correlation function

$$g^{(2)}(\tau) = G^{(2)}(\tau) / |G^{(1)}(0)|^2 = \langle I(t) I(t+\tau)\rangle / I^2(t)$$

Ideal coherent light $E = E_0 e^{-i\omega t + i\varphi}$

$$I(t) = I(t+\tau) = |E_0 e^{-i\omega t + i\varphi}|^2 = |E_0|^2$$

$$g^{(2)}(\tau) = 1 \quad \text{for any } \tau$$

Thermal light $E(t) = \sum_{i=1}^{N} E_i(t)$

$$\langle E^*(t) E^*(t+\tau) E(t+\tau) E(t)\rangle = \sum_{i=1}^{N} \langle E_i^*(t) E_i^*(t+\tau) E_i(t+\tau) E_i(t)\rangle +$$

$$+ \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \left\{ \langle E_i^*(t) E_j^*(t+\tau) E_j(t+\tau) E_i(t)\rangle + \langle E_i^*(t) E_j^*(t+\tau) E_i(t+\tau) E_j(t)\rangle \right\}$$

all other term vanish because of the random phase

$$= N\langle E_i^*(t) E_i^*(t+\tau) E_i(t+\tau) E_i(t)\rangle + N(N-1)\left\{ \langle E_i^*(t) E_i(t)\rangle^2 + \langle E_i^*(t) E_i(t+\tau)\rangle^2 \right\}$$

The first term is negligible since $N \ll N^2$

$$\langle E_i^*(t) E_i(t)\rangle = G^{(1)}(0) = I$$

$$\langle E_i^*(t) E_i(t+\tau)\rangle = G^1(\tau)$$

$$G^{(2)}(t) \approx N^2 (G_i^{(1)}(0) + G^{(1)}(\tau))$$

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2$$

For the thermal light

$$g^{(1)}(\tau) = e^{-\tau/\tau_0}$$

$$g^{(2)}(\tau) = 1 + e^{-2\tau/\tau_0}$$

$$g^{(2)}(0) = 2$$