Coherent state → a pure state of a harmonic oscillator ~~that~~ with mean energy equal to the classical energy

$$\left.\begin{array}{l}\langle\psi|\hat{q}|\psi\rangle = q_c \\ \langle\psi|\hat{p}|\psi\rangle = p_c\end{array}\right\} \text{classical analogs}$$

$$H_c = \frac{p_c^2}{2} + \frac{\omega^2 q_c^2}{2} = \frac{1}{2}\left[\langle\psi|\hat{p}|\psi\rangle^2 + \omega^2\langle\psi|\hat{q}|\psi\rangle^2\right]$$

$$H_c = \hbar\omega \langle\psi|\hat{a}^+|\psi\rangle\langle\psi|\hat{a}|\psi\rangle$$

Corresponding QM average energy

$$\langle\hat{H}\rangle = \hbar\omega\langle\psi|\hat{a}^+\hat{a}|\psi\rangle \quad (+\tfrac{1}{2}\hbar\omega)$$

One can show that for these to values to be equal, $|\psi\rangle$ must be an eigenstate of the annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Coherent states are the eigenstates of the annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

If $|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \Rightarrow \hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}|n-1\rangle$

$$\alpha|\alpha\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle$$

$$c_{n+1}\sqrt{n+1} = \alpha c_n$$

$$\Rightarrow c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$$

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \qquad \langle\alpha|\alpha\rangle = 1 = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = e^{|\alpha|^2}$$

since $\langle n|n'\rangle = \delta_{nn'}$

$$|c_0|^2 \cdot e^{|\alpha|^2} = 1 \Rightarrow |c_0| = e^{-|\alpha|^2/2} = c_0$$

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle}$$

Average value of the electric field

$$\langle\alpha|E_x|\alpha\rangle = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\underbrace{\langle\alpha|\hat{a}|\alpha\rangle}_{\alpha|\alpha\rangle} e^{ikz-i\omega t} + \underbrace{\langle\alpha|\hat{a}^+|\alpha\rangle}_{\alpha^*\langle\alpha|} e^{-ikz+i\omega t}\right)$$

$$= \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\alpha e^{ikz-i\omega t} + \alpha^* e^{-ikz+i\omega t}\right) =$$

$$\alpha = |\alpha| e^{i\varphi} \qquad \alpha^* = |\alpha| e^{-i\varphi}$$

$$= 2|\alpha|\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \cos(kz-\omega t+\varphi)$$

Total energy $\frac{1}{2}\int \varepsilon_0 |E|^2 dV = \underbrace{|\alpha|^2}_{\text{"average" number of photons}} \hbar\omega$

$$\langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^+\hat{a}|\alpha\rangle = |\alpha|^2$$

Photon number fluctuations

$$\Delta n = \sqrt{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}$$

$$\langle \hat{n}^2 \rangle = |\alpha|^4 + |\alpha|^2$$

$$\langle \hat{n} \rangle = |\alpha|^2$$

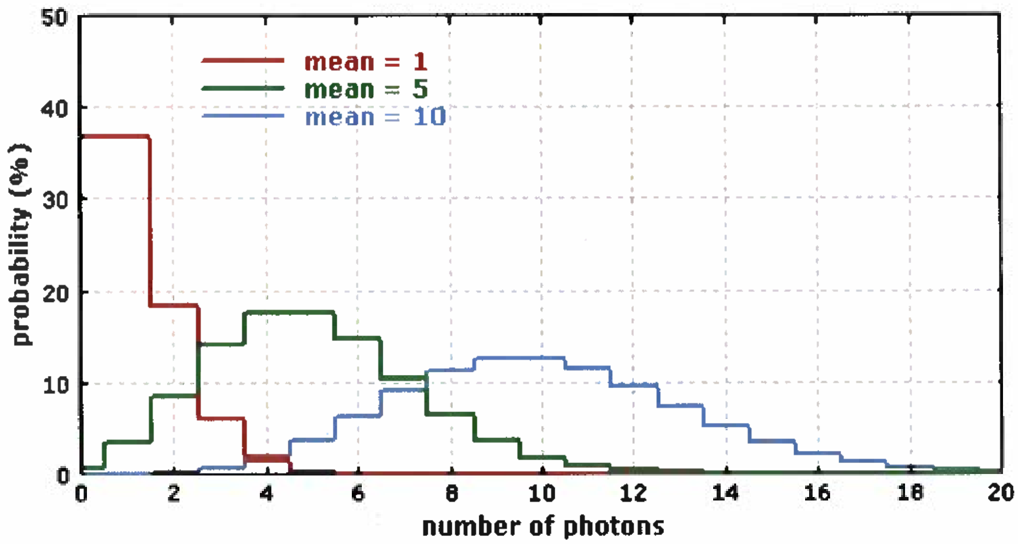
$$\Delta n = |\alpha| = \sqrt{\langle \hat{n} \rangle} = \sqrt{\bar{n}} \quad \text{Poisson process}$$

Photon number distribution

$$p_n = |C_n|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = e^{-\bar{n}} \frac{\bar{n}^n}{n!}$$

$$\frac{\Delta n}{n} = \frac{1}{\sqrt{\bar{n}}} \qquad \text{shot noise}$$

The stronger is the field ($\bar{n} \gg 1$), the more precisely it is defined, closer to the classical description



Photon distribution in coherent states with different mean value of photons $|\alpha|^2$

Electric field fluctuations

$$\langle\alpha|E^2|\alpha\rangle = \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\langle\alpha|\,\hat{a}^2 e^{2(ikz-i\omega t)} + \hat{a}^{+2} e^{-2(ikz-i\omega t)} + \hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a}\,|\alpha\rangle$$

$$= -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left(\alpha^2 e^{2(ikz-i\omega t)} + \alpha^{*2} e^{-2(ikz-i\omega t)} + 1 + 2|\alpha|^2\right) =$$

$$= -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left(\underbrace{2|\alpha|^2\cos 2(kz-\omega t+\varphi) + 2|\alpha|^2}_{4|\alpha|^2\cos^2(kz-\omega t+\varphi)} + 1\right)$$

$$\langle\alpha|E^2|\alpha\rangle = \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left(1 + 4|\alpha|^2\cos^2(kz-\omega t+\varphi)\right)$$

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$$

same fluctuation as in vacuum state

Coherent state is a minimum uncertainty state.

Coherent state is a displaced vacuum state

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle$$

Displacement operator

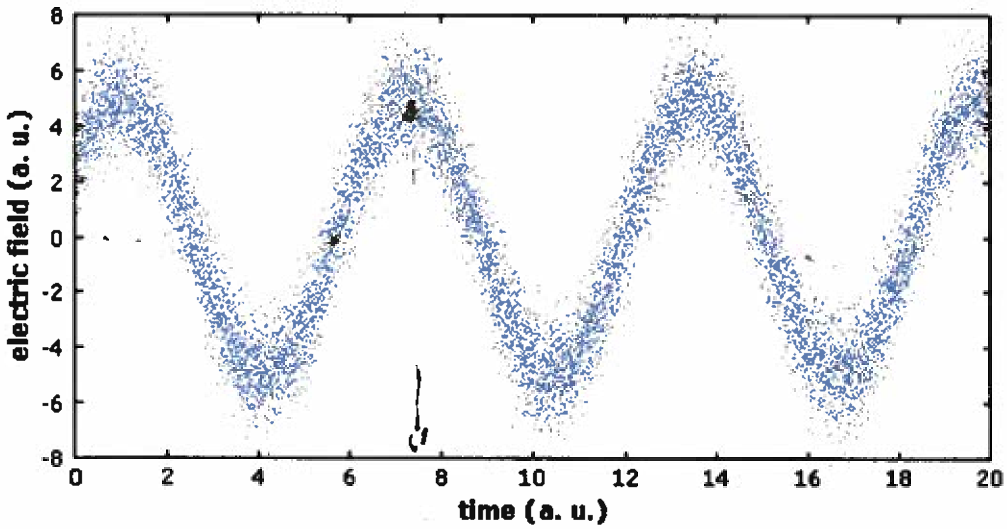
$$\hat{D}(\alpha) = e^{\alpha\hat{a} - \alpha^{*}\hat{a}^{+}}$$

$\hat{D}$ is a unitary operator

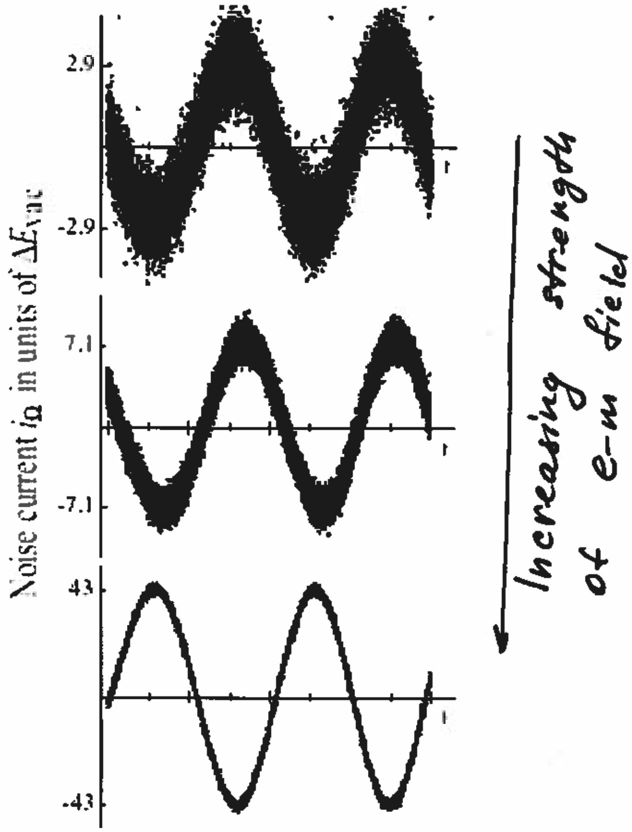
$$\hat{D}(\alpha)\,\hat{D}^{+}(\alpha) = \hat{D}^{+}(\alpha)\,\hat{D}(\alpha) = 1$$

since

$$\hat{D}^{+}(\alpha) = \left(e^{\alpha\hat{a} - \alpha^{*}\hat{a}^{+}}\right)^{+} = e^{\alpha^{*}\hat{a}^{+} - \alpha\hat{a}} = \hat{D}(-\alpha)$$



Coherent state = "fuzzy" electromagnetic wave



Since the uncertainty stays the same as amplitude grows, its effect becomes less and less noticable.

Correspondingly

$$\hat{n}\,(\hat{a}^+|n\rangle) = \hat{a}^+\hat{a}\,(\hat{a}^+|n\rangle) = \hat{a}^+(\hat{a}^+\hat{a}+1)|n\rangle = (n+1)\,\hat{a}^+|n\rangle$$

$\hat{a}^+|n\rangle$ is an eigenstate of $|n\rangle$ with eigenvalue $(n+1)$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \text{[one extra photon after } \hat{a}^+\text{]}$$

Quadrature operators

Electric field

$$E_x(z,t) = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left(\hat{a}e^{-i\omega t} + \hat{a}^+e^{i\omega t}\right)\sin kz$$

Such written $E_x(z,t)$ is a complex operator, so unphysical

$$E_x(z,t) = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left[\hat{a}\,(\cos\omega t - i\sin\omega t) + \hat{a}^+(\cos\omega t + i\sin\omega t)\right] \times \sin kz =$$

$$= \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left[(\hat{a}+\hat{a}^+)\cos\omega t - i(\hat{a}-\hat{a}^+)\sin\omega t\right] \times \sin kz =$$

$$= 2\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left[\hat{X}_1\cos\omega t + \hat{X}_2\sin\omega t\right] \times \sin kz$$

$$\left.\begin{aligned} \hat{X}_1 &= \frac{1}{2}(\hat{a}+\hat{a}^+) \\ \hat{X}_2 &= \frac{1}{2i}(\hat{a}-\hat{a}^+) \end{aligned}\right] \text{Quadrature operators}$$

They represent "real" and "imaginary" parts of e-m field, oscillating with frequency ω with $\pi/2$ phase lag between each other.

We can also make connections with canonical position and momentum operators for e-m field

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p})$$

$$\hat{a}^{+} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p})$$

Thus

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^{+}) = \frac{\omega}{\sqrt{2\hbar\omega}}\,\hat{q} \quad \text{(like electric field)}$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^{+}) = \frac{1}{\sqrt{2\hbar\omega}}\,\hat{p} \quad \text{(like magnetic field)}$$

Thus, two quadratures are expected to behave as quantum position and momentum → not commute!

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$$

That implies $\langle(\Delta\hat{X}_1)^2\rangle\langle(\Delta\hat{X}_2)^2\rangle \geq \frac{1}{16}$

Number state $\langle n|\hat{X}_{1,2}|n\rangle = 0$

$$\langle(\Delta\hat{X}_1)^2\rangle = \langle n|\,\hat{X}_{1,2}^2\,|n\rangle = \frac{1}{4}\langle n|\,\hat{a}^2 + \hat{a}^{+2} + \hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+}|n\rangle$$

$$= \frac{1}{4}\langle n|\,2\hat{a}^{+}\hat{a} + 1\,|n\rangle = \frac{1}{4}(2n+1)$$

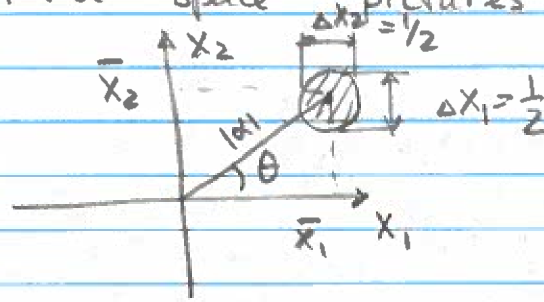
Vacuum state $(n=0)$ $\langle\Delta X_1^2\rangle\langle\Delta X_2^2\rangle = \frac{1}{16}$

min uncertainty state

Coherent state is also a min uncertainty state

$$\langle\Delta X_1^2\rangle = \langle\Delta X_2^2\rangle = \frac{1}{4}$$

Phase-space pictures

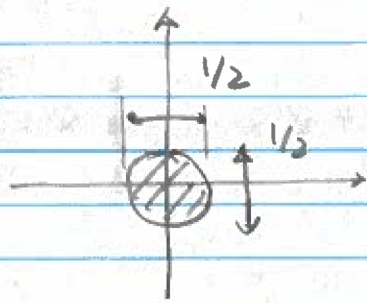


Coherent state

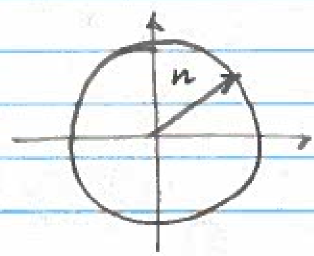
$$\bar{X}_1 = \langle \hat{X}_1 \rangle = \langle \alpha | \tfrac{1}{2}(\hat{a} + \hat{a}^{+}) | \alpha \rangle = \frac{\alpha + \alpha^*}{2} = \mathrm{Re}(\alpha)$$
$$= |\alpha| \cos\theta$$

$$\bar{X}_2 = \langle \hat{X}_2 \rangle = \frac{\alpha - \alpha^*}{2i} = \mathrm{Im}(\alpha) = |\alpha| \sin\theta$$

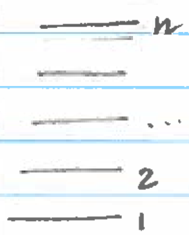
Vacuum state



Number state



Thermalized harmonic oscillator



Classical case – Boltzman distribution

$$P_n = \frac{e^{-E_n/k_BT}}{\underbrace{\sum_n e^{-E_n/k_BT}}_{Z}} \qquad E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$

Quantum case:

$$\hat{\rho}_{th} = \frac{e^{-\hat{H}/k_BT}}{\underbrace{\mathrm{Tr}\left[e^{-\hat{H}/k_BT}\right]}_{Z}} \qquad \hat{H} = \hbar\omega\left(\hat{n}+\tfrac{1}{2}\right) = \hbar\omega\left(\hat{a}^+\hat{a}+\tfrac{1}{2}\right)$$

$$Z = \mathrm{Tr}\left[e^{-\hat{H}/k_BT}\right] = \sum_{n=0}^{\infty}\langle n|e^{-\hat{H}/k_BT}|n\rangle =$$

$$= \sum_{n=0}^{\infty} e^{-\hat{E}_n/k_BT} = \sum_{n=0}^{\infty} e^{-\hbar\omega(n+1/2)/k_BT} =$$

$$= \frac{e^{-\frac{\hbar\omega}{2k_BT}}}{1-e^{-\hbar\omega/k_BT}}$$

Average number of photons

$$\langle\hat{n}\rangle = \mathrm{Tr}\langle\hat{n}\hat{\rho}_{th}\rangle = \sum_{n=0}^{\infty}\langle n|\hat{n}\hat{\rho}_{th}|n\rangle =$$

$$= \sum_{n=0}^{\infty} n\underbrace{\langle n|\hat{\rho}_{th}|n\rangle}_{P_n} = \frac{1}{Z}\sum_{n=0}^{\infty} n e^{-E_n/k_BT} =$$

$$= \frac{e^{-\frac{\hbar\omega}{2k_BT}}}{Z}\sum_{n=0}^{\infty} n\cdot e^{-\frac{\hbar\omega n}{2k_BT}}$$

$$\sum_{n=0}^{\infty} n e^{-\beta n} = \frac{e^{-\beta}}{(1-e^{-\beta})^2}$$

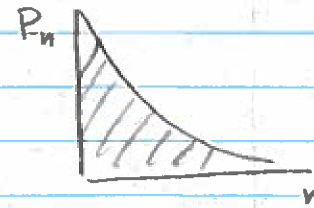
$$\bar{n} = \frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1} \longrightarrow \begin{cases} \hbar\omega \ll k_BT \text{ (thermal radiation)} \\ \bar{n} \approx k_BT/\hbar\omega \\ \hbar\omega \gg k_BT \text{ (optical ranges for room temperature)} \\ \bar{n} \approx e^{-\hbar\omega/k_BT} \end{cases}$$

Bose-Einstein statistics

Since $e^{-\frac{\hbar\omega}{k_BT}} = \frac{\bar{n}}{1+\bar{n}}$

we can rewrite

$$\hat{\rho}_{th} = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|$$



$$P_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}}$$

$$\Delta n = \sqrt{\bar{n} + \bar{n}^2}$$

$$\frac{\Delta n}{n} = \sqrt{1 + \frac{1}{\bar{n}}} \xrightarrow{\bar{n} \gg 1} 1$$

$$\xrightarrow{\bar{n} \ll 1} \infty$$

Total energy density

density of states

$$u(\omega) = \hbar\omega\, \bar{n}(\omega) \cdot \rho(\omega)$$

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/k_BT} - 1}$$

Planck's radiation law

One can use it to derive all the laws describing black-body radiation.