Some standard notation

Last time, we talked about decay rates for atomic populations and coherences

$$\frac{\partial \rho_{aa}}{\partial t} = -\gamma_a \rho_a + \dots \qquad \frac{\partial \rho_{bb}}{\partial t} = -\gamma_b \rho_{bb} + \dots$$

$$\frac{\partial \rho_{ab}}{\partial t} = -\gamma_{ab} \rho_{ab} + \dots$$

Typically, two relaxation times are assigned to an atomic transition (for a given experiment)

T_1 - $1/e$ decay time of the population difference

T_2 - ——— " ——— of the induced dipole

This notation originated from describing atomic spins in NMR.

For a pure radiative decay (spontaneous emission) $\quad T_1 \simeq \frac{1}{\gamma_a} + \frac{1}{\gamma_b}$ (if $\gamma_a \gg \gamma_b$)

$$\gamma_{ab} = \frac{\gamma_a + \gamma_b}{2} \qquad T_2 = \frac{1}{\gamma_{ab}} = \frac{2}{\gamma_a + \gamma_b}$$

In case of additional dephasing $\quad \gamma_{ab} = \frac{\gamma_a + \gamma_b}{2} + \gamma_{dph}$

$$T_2 = \frac{1}{\gamma_{ab}} = \frac{1}{\frac{\gamma_a + \gamma_b}{2} + \gamma_{dph}}$$

In systems with many emitters, it is also common to introduce T_2^* - decoherence rate that includes inhomogeneity within the ensemble, when we take the ensemble average.

Absorption and amplification of light in a resonant atomic medium

Wave equation

$$-\nabla^2 \vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \underbrace{-\mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}}_{\text{effect of the atoms}} \qquad \vec{P} = N\langle \vec{d}\rangle$$

Assuming $\vec{d} \parallel \vec{E}$ $\qquad \hat{d} = \langle a| -er |b\rangle \left(|a\rangle\langle b| + |b\rangle\langle a|\right)$

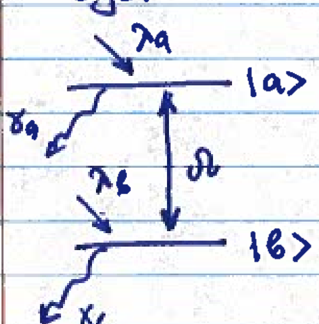
Slowly varying amplitude and phase approximation

$$\frac{\partial \mathcal{E}_0}{\partial t} + \frac{1}{c}\frac{\partial \mathcal{E}_0}{\partial t} = \frac{ik}{2\varepsilon_0} P_0 \qquad P_0 = \wp_{ab}\,\tilde{\rho}_{ab}$$

Thus, to calculate changes in EM field caused by atoms, we need to find $\tilde{\rho}_{ab}$ (off-diagonal density matrix element)

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}] + \left\langle \begin{matrix}\text{decay terms} \\ \text{repumping terms}\end{matrix} \right\rangle$$

Let's consider ~~the~~ ~~most~~ a general two-level system



γ_a, γ_b – decay rates of $|a,b\rangle$
λ_a, λ_b – pumping rates — " —

γ_{ab} – decoherence rate of ρ_{ab}
$\Delta = \omega - \omega_{ab}$

Maxwell-Bloch equations

$$\begin{cases} \dot{\rho}_{ab} = -(\gamma_{ab} - i\Delta)\rho_{ab} + i\Omega(\rho_{aa} - \rho_{bb}) \\ \dot{\rho}_{aa} = \lambda_a - \gamma_a \rho_{aa} - i\Omega(\rho_{ba} - \rho_{ab}) \\ \dot{\rho}_{bb} = \lambda_b - \gamma_b \rho_{bb} + i\Omega(\rho_{ba} - \rho_{ab}) \end{cases}$$

The steady-state solution $\frac{\partial \rho_{ij}}{\partial t} = 0$

$$\rho_{ab} = i\Omega \frac{\rho_{aa} - \rho_{bb}}{\gamma_{ab} - i\Delta}$$

$$\rho_{ba} - \rho_{ab} = \Omega(\rho_{aa} - \rho_{bb}) \left[-\frac{i}{\gamma_{ab} + i\Delta} - \frac{i}{\gamma_{ab} + i\Delta} \right]$$

$$= -i\Omega(\rho_{aa} - \rho_{bb}) \frac{2\gamma_{ab}}{\gamma_{ab}^2 + \Delta^2}$$

$$\rho_{aa} = \frac{1}{\gamma_a} (\lambda_a - i\Omega(\rho_{ba} - \rho_{ab}))$$

$$\rho_{bb} = \frac{1}{\gamma_b} (\lambda_b + i\Omega(\rho_{ba} - \rho_{ab}))$$

No applied EM field $(\Omega = 0)$

$$\rho_{aa}^{(0)} - \rho_{bb}^{(0)} = \frac{\lambda_a}{\gamma_a} - \frac{\lambda_b}{\gamma_b} = \Delta\rho_0$$

unsaturated population difference

~~Rea~~ With the applied field

$$\rho_{aa} - \rho_{bb} = \Delta\rho_0 - 2\Omega^2 \underbrace{\left(\frac{1}{\gamma_a} + \frac{1}{\gamma_b}\right)}_{T_1} \underbrace{\frac{\gamma_{ab}}{\gamma_{ab}^2 + \Delta^2}}_{T_2 L(\Delta)} (\rho_{aa} - \rho_{bb})$$

$$L(\Delta) = \frac{\gamma_{ab}^2}{\gamma_{ab}^2 + \Delta^2}$$

Lorentzian function contains the effect of the detuning

$$\rho_{aa} - \rho_{bb} = \Delta\rho_0 - 2\Omega^2 T_1 T_2 L(\Delta)(\rho_{aa} - \rho_{bb})$$

$$\rho_{aa} - \rho_{bb} = \frac{\Delta\rho_0}{1 + \underbrace{2\Omega^2 T_1 T_2 L(\Delta)}_{\text{saturation}}}$$

The stronger is the field, the smaller is the population difference

For $\Delta = 0$ (on resonance)

$$\rho_{aa} - \rho_{bb} = \frac{\Delta\rho_0}{1 + 2\Omega^2 T_1 T_2} = \frac{\Delta\rho_0}{1 + I/I_s}$$

I – intensity of the laser field $I = c\varepsilon_0 |\mathcal{E}_0|^2$

I_s – saturation intensity $(2\Omega^2 T_1 T_2 = 1)$

$$I_s = \frac{c\varepsilon_0 (2\hbar\,\wp_{ab})^2}{T_1 T_2} \qquad \Omega = \frac{\wp_{ab}\mathcal{E}_0}{\hbar}$$

$$\rho_{ab} = i\Omega \frac{\rho_{aa} - \rho_{bb}}{\gamma_{ab} \mp i\Delta} = i \frac{\Omega}{\gamma_{ab} \mp i\Delta} \frac{\Delta\rho_0}{1 + I/I_s\, L(\Delta)}$$

Induced polarization $P = \wp_{ab}\,\rho_{ab}$

$$P_0(z) = -i \frac{\wp_{ab}^2}{\hbar} \frac{\Delta\rho_0}{1 + I/I_s\, L(\Delta)} \frac{1}{\gamma_{ab} \mp i\Delta} \cdot \mathcal{E}_0(z)$$

Introducing the susceptibility again $P_0 = \varepsilon_0 \chi\, \mathcal{E}_0$

$$\chi = -i \frac{\wp_{ab}^2}{\varepsilon_0\hbar} \frac{\Delta\rho_0}{1 + I/I_s L(\Delta)} \frac{1}{\gamma_{ab} \mp i\Delta} = \frac{-i\gamma_{ab} + \Delta}{\Delta^2 + \gamma_{ab}^2} \frac{\wp_{ab}^2}{\varepsilon_0\hbar} \frac{\Delta\rho_0}{1 + I/I_s L(\Delta)}$$

The absorption coefficient $\mathcal{E}(z) = e^{-\alpha z}\, \mathcal{E}_0(0)$

$$\alpha = \frac{k}{2}\chi'' = -\frac{\wp_{ab}^2}{2\varepsilon_0\hbar} k \frac{\Delta\rho_0}{1 + I/I_s L(\Delta)} \left.\frac{\gamma_{ab}}{\gamma_{ab}^2 + \Delta^2}\right] \frac{L(\Delta)}{\gamma_{ab}}$$

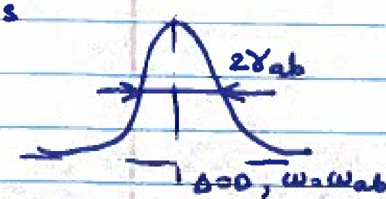
$$\alpha = \underbrace{-\frac{\wp_{ab}^2}{2\varepsilon_0\hbar} k\, \Delta\rho_0 \frac{1}{\gamma_{ab}}}_{\alpha_0} \frac{L(\Delta)}{1 + I/I_s L(\Delta)}$$

The absorption coefficient depends on the light intensity

(welcome to nonlinear optics!)

Very weak optical field $\quad I \ll I_s$

$$d(\Delta) \simeq d_0 L(\Delta) = d_0 \frac{\gamma_{ab}^2}{\gamma_{ab}^2 + \Delta^2}$$



same lineshape as for the classical oscillator model!

$$d_0 = -\frac{k}{2} \frac{p_{ab}^2 \, \Delta\rho_0}{\varepsilon_0 \hbar \gamma_{ab}} = -\frac{k}{2} \frac{p_{ab}^2}{\varepsilon_0 \hbar \gamma_{ab}} \left(\rho_{aa}^{(0)} - \rho_{bb}^{(0)}\right)$$

Resonant <u>unsaturated</u> absorption* $\quad$ *(most of the time)

$$\mathcal{E}_0(z) = \mathcal{E}_0(0)\, e^{-d_0 z} \qquad (\text{for } \Delta = 0)$$

If $d_0 > 0$, $\mathcal{E}_0(z)$ weakens as it propagates through the atoms $\longrightarrow$ absorption

$d_0 > 0 \qquad \Delta\rho_0 < 0 \qquad \rho_{aa}^{(0)} < \rho_{bb}^{(0)}$

no population inversion

If $d_0 < 0 \qquad \mathcal{E}_0(z) = \mathcal{E}_0(0)\, e^{|d_0| z}$ — field grows

gain medium

Requires population inversion $\qquad \rho_{aa}^{(0)} > \rho_{bb}^{(0)}$

$$\lambda_a / \gamma_a > \lambda_b / \gamma_b$$

Unless we are talking about some exotic quantum effects, the population inversion is required for amplification (will get back to that when discussing lasers)

Strong EM field $I \geq I_s$

nonlinear susceptibility $\chi(I)$

$$\frac{\partial \mathcal{E}}{\partial z} = -\alpha \mathcal{E} \qquad I = c\varepsilon_0 \frac{|\mathcal{E}|^2}{2}$$

$$\frac{\partial I}{\partial z} = -2\alpha I = -2\alpha_0 I \frac{L(\Delta)}{1 + I/I_s\, L(\Delta)}$$

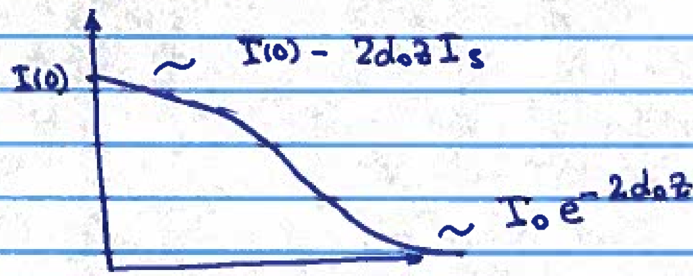
Solving this equation

$$\ln\left(\frac{I(z)}{I(0)}\right) + \frac{1}{I_s}\left(I(z) - I(0)\right)L(\Delta) = -2\alpha_0 L(\Delta) z$$

Exact solution, but not terribly illuminating

$I \gg I_s$ and $L(\Delta) \sim 1$

$$I(z) \simeq I(0) - 2\alpha_0 z I_s$$



Detuning dependence

Unsaturated $I(\Delta, z) = I_0\, e^{-\alpha_0 z \frac{\gamma_{ab}^2}{\gamma_{ab}^2 + \Delta^2}}$