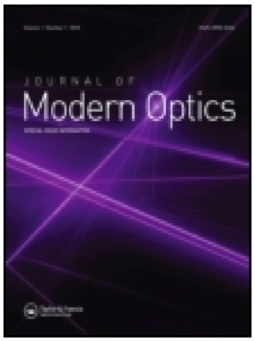

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Quantum entanglement criteria

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We review the main criteria used to detect entanglement in quantum systems. The main properties of each criteria are summarized depending on whether the criteria provides a sufficient or necessary condition, whether it involves density matrix or operators, or if the criteria is phase sensitive. We show that several criteria have much in common and they could be related mathematically. We also discuss the features of entanglement which are useful in quantum information technology.

Keywords: quantum optics; entanglement; quantum information; correlation

1. Introduction to the nature of entanglement

Quantum entanglement is a property of a quantum mechanical system. The mystery of entanglement was sparked by Einstein, Podolsky and Rosen in a debate, which was later named as the EPR paradox [1]. Later, Schrödinger developed a thought experiment, which bears the concept of Schrödinger's cat [2]. The quantum entanglement is the most puzzling feature of quantum mechanics [3]. Entanglement happens when a system of two quantum particles are linked together and might become entangled even after the particles are spatially separated.

Some physicists said that entanglement is a strange phenomenon because separated particles are linked intrinsically [4] by some unphysical forces. According to Albert Einstein, this is a 'spooky action at a distance' [5] that violates the theory of special relativity. This leads to the search for hidden variables in quantum mechanics. This concept can be interpreted as particles separated at a distance linked together instantaneously and influenced by one another. The separated particles become entangled when there is interaction between the particles even without physical contact [6]. This phenomena leads to the notions of inseparability and nonlocality of quantum particles. The entanglement demonstrates a strong quantum correlation where the linked particles share mutual information contained in each particle, such as a two identical and indistinguishable binary system [7].

The mystery of entanglement is usually connected to nonlocality which causes a lot of confusion. Experiments on entanglement show that nonlocality is an intrinsic property of the quantum world. The Schrödinger's cat thought experiment shows that the possible outcome can help to predict the superposition of two coherent states [8]. The result of the possible outcomes, i.e. whether the cat's state is dead or alive corresponds to the collapse of entanglement in the coherent state [8]. The two possible results of the cat's state show that an entangled state must be a superposition of distinguishable states.

Although quantum entanglement is often characterized as weird, it could be useful for transmitting information [6] and for quantum information processing. It is the key ingredient of quantum information and quantum communication. It also serves as an essential basis for quantum computing and cryptography [6]. An entangled system is a necessary tool for information transfer because the process involves information sharing without any hidden interception. This is very useful for confidential communication especially in the field of military and defense where the conveyed information will be converted into secret codes [9]. Both parties use strong correlation for transmitting private messages that can detect any attempt of an eavesdropper to invade the privacy of their communication. Quantum cryptography is another asset of quantum mechanics in information theory [10]. Using pairs of entangled states in quantum cryptography enables information to be converted into shared secret keys [11]. It prevents unintended information leakage [12] since all the information in the message is contained in the quantum correlations between the entangled pairs. Unfortunately, there are technological challenges in protecting the valuable information against decoherence in the environment, which is necessary, for example, in superdense coding and quantum teleportation.

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Quantum entanglement appears everywhere in the microscopic world, it can happen in any interaction between quantum particles such as atoms, photons, electrons or molecules. There are optical phenomena that exhibit entanglement or nonclassical properties. The presence of entanglement of particles has been predicted by many active works using parameters like density operator, photon number statistics and quantum states. Entanglement can be identified, for examples as perfect squeezing of light or photon antibunching. Other popular quantities are entropy, positive partial transposition and two mode squeezing [2] that may involve many-particle systems and thermodynamics.

The aim of this paper is to review the criteria of detecting entanglement of a system theoretically. From previous studies, different methods were discovered, with each having various entanglement criteria in terms of several parameters. This paper is divided into three sections. In Section 2, we review all entanglement criteria which have been discovered and the relationships among these criteria to exhibit entanglement. This section also contains the conditions that should be observed in order to detect the entanglement. Section 3 discusses the quantities that can quantify entanglement that are important in quantum information science.

2. Quantum entanglement criteria

There are various criteria for describing entanglement and nonclassical correlation. In each criteria, entanglement may be detected in a system if the system satisfies certain conditions. In this section, we will review twelve criteria of entanglement, along with their conditions in order to detect entanglement in a system. A list of these criteria is shown in Figure 1. The relationship between these criteria will also be discussed (Figure 2), in order to obtain better understanding of entanglement.

2.1. *Entropy*

In this method, the density operator is an important parameter for describing the entanglement of a system. The measured entropy is referred to as the von Neumann entropy given by $S(\rho) = -k_B Tr(\rho \ln \rho)$ (where k_B is the Boltzmann constant) and it is connected to the thermodynamic property [8]. A system that can use the entropy criteria to describe entanglement can be an interaction between a pair of particles. Here, two systems A and B are considered as correlated when system $A(B)$ projects to system $B(A)$. The projection of the system will then produce a new state which is known as the mixed state [13]. A mixed state contains more shared information compared to a pure state. The von Neumann entropy, $S(\rho)$ enables us to describe entanglement based on the quantity in the framework of quantum information theory [14]. The classical correlation entropy, $S(A:B)$, which was introduced by [14] and [15], is denoted as

$$S(A:B) = S(A) + S(B) - S(A, B). \quad (1)$$

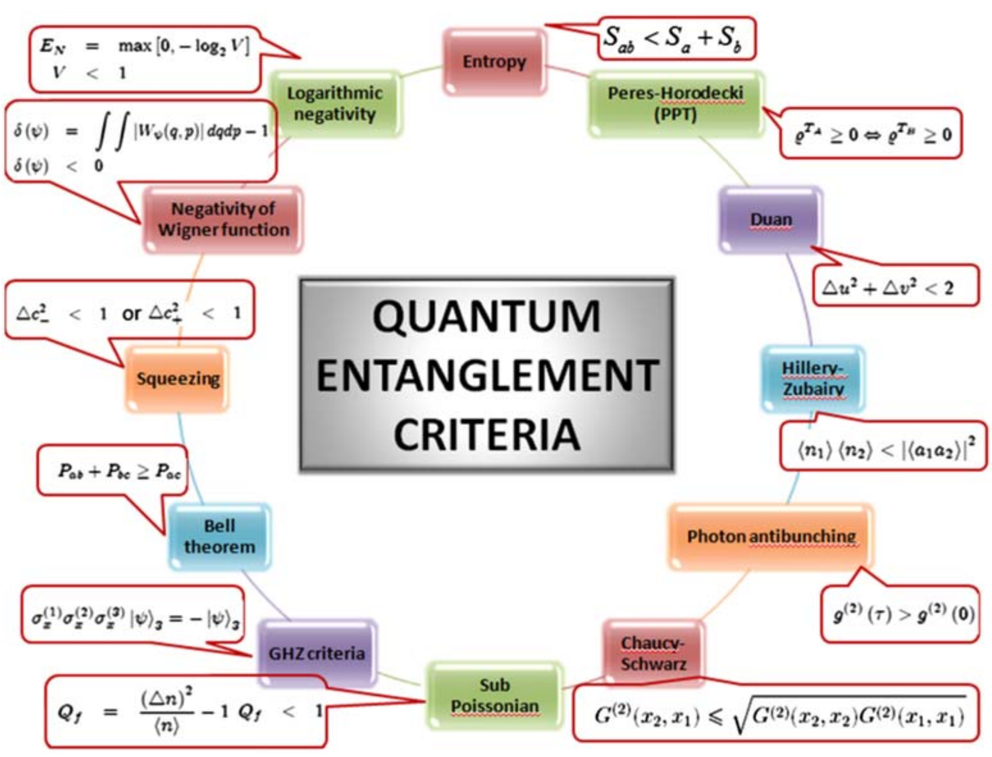

Figure 1. Various entanglement criteria widely used to detect entanglement.

CHECKLISTS PROPERTIES OF QUANTUM ENTANGLEMENT

Properties / Criteria	Necessary condition	Sufficient condition	Density operator	No. of Photon	Correlation	Phase sensitive
Entropy	✓		✓			
Peres Horodecki	✓		✓			
Duan		✓		✓	✓	✓
Hillery-Zubairy		✓		✓	✓	✓
Antibunching	✓			✓	✓	✓
Chaucy-Schwarz	✓			✓	✓	
Sub Poissonian	✓			✓	✓	✓
GHZ		✓		✓		
Bell's theorem	✓			✓		
Squeezing	✓			✓	✓	✓
Negative Wigner function		✓	✓			✓
Logarithmic Negativity		✓		✓		✓

Figure 2. Properties of each entanglement criteria.

The entropy criteria shows that the total entropy of two systems A and B should be greater than the joint entropy. This is a necessary condition for entanglement. The system should be entangled when it satisfies this necessary condition of quantum entanglement. The joint von Neumann entropy $S(A, B)$ is a measure of the quantum correlation of a system. If the subsystems A and B are statistically uncorrelated, the classical correlation entropy $S(A : B) = 0$. In contrast, if $S(A : B) > 0$ or $S(A) + S(B) > S(A, B)$ the system is said to be statistically correlated [15] or entangled.

2.2. Peres–Horodecki

The second method for describing entanglement and inseparability is also known as positive partial transpose $\tilde{\rho}$ of the density matrix. It enables the prediction of quantum entanglement of a system [16]. For a quantum system with two subsystems, A with N dimensions and B with M dimensions, the composite density operator is expanded into

$$\rho = \sum_{ij}^{N} \sum_{kl}^{M} \rho_{ij,kl} \left| i \right\rangle \left\langle j \right| \otimes \left| k \right\rangle \left\langle l \right| . \tag{2}$$

In order to transpose the system, the two separable density matrices are summed up into the direct product Equation (2). The transpose of the density matrix is non-negative matrices with unit trace where the eigenvalue should also always be positive [17]. In a similar manner, the partial transpose is identified as a mirror reflection in phase space [16]. The Peres–Horodecki criterion is not complete without involving the projection of a subsystem into another subsystem due to the fundamental problem of operation which arises from a separable state. In this case, the separability of a system can be analyzed according to [18]. The trace of any density matrix should be positive for any projection, $\mathrm{Tr}\rho \geq 0$ with $\mathrm{Tr}\rho = 1$. Prior to this, the projection was referred to the positive mapping of a set of operators $\{A_1\}$ and $\{A_2\}$ which act on Hilbert spaces H_1 and H_2, respectively. The set of positive operators can be determined as $\Lambda\,(A) \geq 0$, for a set of operators $\{A\} \geq 0$. The positive mapping operator, Λ applied to linear positive mapping $\Lambda \in L\,(A_1, A_2)$ will produce a completely positive results. Upon employing positive mapping to the tensor product of the density operator, $(\Lambda\rho) \otimes \tilde{\rho} \geq 0$, an inseparable state is obtained. Therefore, the state is recognized as an inseparable state when a positive map exists [18].

For a given separable density matrix Equation (2), we can easily calculate the partial transpose with respect to one subsystem,

$$\rho^{T_A} = \sum_k p_k \left(\rho_k^A\right)^T \otimes \rho_k^B \tag{3}$$

$$= \sum_k p_k \tilde{\rho}_k^A \otimes \rho_k^B \geq 0. \tag{4}$$

The discovery of this criteria proves that the partial transpose of a two separable density matrix leads to a positive eigenvalue and overcomes the inseparability of the subsystem. If the eigenvalue is negative, the state may still be entangled if it satisfies the sufficient condition of separability. In simpler words, the system is inseparable when the transpose of any density matrix is always positive, that is $\rho^{T_A} \geq 0$ or $\rho^{T_B} \geq 0$ [2].

Both criteria, i.e. the entropy entanglement criteria and the Peres–Horodecki criteria are very useful for the detection of entanglement in a bipartite quantum state that involves two systems. Moreover, these criteria are categorised as the necessary conditions of entanglement (Figure 2).

2.3. Squeezing

Due to the effect of quantization, a relationship is established for photons as harmonic oscillators in quantum mechanics [19]. A light wave is known to be in a coherent state if it can be described as a classical electromagnetic wave. The process of quantization introduces the uncertainty principle between the photon number and its phase. The properties of nonclassical light corresponds to a squeezed state while a coherent state is a minimum uncertainty state having two equal quadratures.

Squeezing of light is a quantum optical phenomena which corresponds to the entanglement in the system. One way to detect entanglement is by identifying the squeezing. The squeezing of light arises from the correlation between two orthogonal quadratures formed by the annihilation operator $\hat{a}$ and creation operator $\hat{a}^{\dagger}$ as given by

$$\hat{c}_{+} = \frac{1}{2}\left(\hat{a}^{\dagger} + \hat{a}\right) \text{ and } \hat{c}_{-} = \frac{1}{2\mathrm{i}}\left(\hat{a}^{\dagger} - \hat{a}\right). \quad (5)$$

The quadrature operators should satisfy the commutation relation, $\left[\hat{c}_{+}, \hat{c}_{-}\right] = \mathrm{i}/2$ and the product of the quadrature variances should satisfy the inequality, $\langle(\triangle\hat{c}_{+})^{2}\rangle\langle(\triangle\hat{c}_{-})^{2}\rangle \geq \frac{1}{16}$ [8]. The squeezing of a field can be compared to the squeezing in position and the momentum of a quantized harmonic oscillator. The variances of two quadratures with identical value $\triangle\hat{c}_{+} = \triangle\hat{c}_{-} = \frac{1}{2}$ correspond to the minimum uncertainty state [19]. Perfect squeezing can be identified when it satisfies either of the conditions [20]

$$\langle\triangle\hat{c}_{\mp}^{2}\rangle \ll 1 \text{ with } \langle\triangle\hat{c}_{\pm}^{2}\rangle \approx 0. \quad (6)$$

These conditions demonstrate that a squeezed state, $\hat{c}_{+}$ will force $\hat{c}_{-}$ to be in a reduced noise state, and vice versa. The process of reducing the uncertainty in amplitude would expand the uncertainty in phase [21]. The squeezing of light is connected with the basis of entanglement criteria.

The squeezing of photon number is easily determined according to the photon number operator $\hat{n} = \hat{a}^{\dagger}\hat{a}$ [8] described as,

$$\left\langle(\triangle\hat{n})^{2}\right\rangle = \langle\bar{n}\rangle - \langle\bar{n}\rangle^{2} + \left\langle\hat{a}^{\dagger 2}\hat{a}^{2}\right\rangle, \quad (7)$$

where $\langle\bar{n}\rangle$ is the average number of photons.

2.4. Sub-Poissonian

There is a close relationship between the squeezing and the sub-Poissonian due to the amplitude of phase squeezing expressed in the sub-Poissonian statistic [19]. The state with sub-Poissonian statistic has a narrower peak in the photon number distribution than the Poissonian statistic [22]. The sub-Poissonian statistic of a field state shows non-classical properties and possibly the entanglement of the field. The sub-Poissonian criteria is defined based on the Mandel Q parameter of the field [8],

$$Q_{f} = \frac{\left\langle(\triangle\hat{n})^{2}\right\rangle}{\langle\bar{n}\rangle} - 1. \quad (8)$$

The sub-Poissonian condition falls in the range $-1 \leq Q_{f} < 0$. This range implies $\langle(\triangle\hat{n})^{2}\rangle/\langle\bar{n}\rangle < 1$, i.e. the photon number fluctuation $\langle(\triangle\hat{n})^{2}\rangle$ is less than the average photon number $\langle\bar{n}\rangle$. This condition may be interpreted as due to the existence of an anticorrelated state. The maximum sub-Poissonian statistic occurs when $Q_{f} = -1$ or $\langle(\triangle\hat{n})^{2}\rangle \approx 0$.

Quantum correlations of photons is related to photon number fluctuation or variance with nonclassical amplitude squeezing through the second-order correlation function

$$g^{(2)}(t) = \frac{\left\langle\hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\right\rangle}{\left\langle\hat{a}^{\dagger}\hat{a}\right\rangle^{2}} = 1 + \frac{\left\langle(\triangle\hat{n})^{2}\right\rangle - \langle\hat{n}\rangle}{\langle\bar{n}\rangle^{2}}. \quad (9)$$

It can be expressed in terms of the variance, Equation (7) and the Q_{f} parameter,

$$g^{(2)}(t) = 1 + \frac{Q_{f}}{\langle\bar{n}\rangle}. \quad (10)$$

Thus, sub-Poissonian ($Q_{f} < 0$) corresponds to $g^{(2)}(t) < 1$. In the limit of large mean photon number the light has second-order coherence $g^{(2)}(t) \rightarrow 1$. This shows that the sub-Poissonian property cannot be found in intense light fields.

2.5. Photon antibunching

Photon antibunching phenomena happens when the statistics of photons is scattered in the time evolution. It corresponds when fewer photon pairs are being detected closer together in time. The correlation of scattered photons is studied using the second-order correlation function of photodetection with respect to time [23]

$$g^{(2)}(\tau) = \frac{\left\langle\hat{a}^{\dagger}(t)\,\hat{a}^{\dagger}(t+\tau)\,\hat{a}(t+\tau)\,\hat{a}(t)\right\rangle}{\left\langle\hat{a}^{\dagger}\hat{a}\right\rangle^{2}}. \quad (11)$$

Based on a photodetection experiment [24], for a coherent state, $g^{(2)}(\tau) = 1$ represents the highly correlated state. In this state, the probability of joint detection coincides with the probability of independent detection. Normally, the correlation of two-photon photodetection will die out, $g^{(2)}(\tau) = 0$ when the time delay approaches infinity, $\tau \rightarrow \infty$ [25], i.e. the joint probability of detecting the second photon decreases with time delay. This situation $g^{(2)}(\tau) < g^{(2)}(0)$ is identified as photon bunching, i.e. two photons tend to be detected simultaneously or after a short time delay.

On the other hand, if $g^{(2)}(\tau) > g^{(2)}(0)$, the joint probability of detecting the second photon increases with time delay. This is known as photon antibunching. Here, $g^{(2)}(\tau) \to 1$ for $\tau \to \infty$, and $g^{(2)}(0) < 1$ implies that there is increased probability of detecting a second photon after a finite time delay, τ [8]. This counterintuitive effect is the result of the quantum nature of light. Thus, photon antibunching is one of the methods to describe the entanglement. A field is said to be entangled if the inequality, $g^{(2)}(\tau) > g^{(2)}(0)$ is satisfied. For the coherent state, $g^{(2)}(\tau) = 1$ represents a classical state. However, for a nonclassical field state, we have $g^{(2)}(0) < 1$ a violation of the classical result. Therefore, the photon antibunching phenomena occur when $g^{(2)}(0) < 1$ and $g^{(2)}(\tau) > g^{(2)}(0)$, implying the presence of entanglement.

2.6. *Cauchy–Schwarz inequality*

We may also use the second-order correlation to determine the entanglement of a system with two modes, $\hat{a}$ and $\hat{b}$. The system is said to be entangled if it satisfies the Cauchy–Schwarz inequality. The expectation value of cross-correlation between two modes $\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle$ is bounded by [26]

$$\left\langle \hat{a}^{\dagger 2} \hat{a}^2 \right\rangle \left\langle \hat{b}^{\dagger 2} \hat{b}^2 \right\rangle \geq \left\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \right\rangle^2 . \tag{12}$$

Two modes have strong nonclassical correlation if the correlation between two modes violates the above Cauchy–Schwarz inequality. The inequality may be expressed in terms of the second-order correlation functions at zero time [22]

$$g_a^{(2)}(0) = \frac{\left\langle \hat{a}^{\dagger 2} \hat{a}^2 \right\rangle}{\left\langle \hat{a}^\dagger \hat{a} \right\rangle^2}, \quad g_b^{(2)}(0) = \frac{\left\langle \hat{b}^{\dagger 2} \hat{b}^2 \right\rangle}{\left\langle \hat{b}^\dagger \hat{b} \right\rangle^2} \tag{13}$$

and the second-order cross-correlation function

$$g_{ab}^{(2)}(0) = \frac{\left\langle \hat{b}^\dagger \hat{a}^\dagger \hat{a} \hat{b} \right\rangle}{\left\langle \hat{a}^\dagger \hat{a} \right\rangle \left\langle \hat{b}^\dagger \hat{b} \right\rangle}. \tag{14}$$

Assuming $\langle \hat{b}^\dagger \hat{a}^\dagger \hat{a} \hat{b} \rangle = \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle$ the Cauchy–Schwarz inequality can be expressed as

$$g_a^{(2)}(0)\, g_b^{(2)}(0) \geq \left[g_{ab}^{(2)}(0) \right]^2 . \tag{15}$$

The entanglement in a system can be identified if the above inequality is satisfied.

Optical phenomena like sub-Poissonian, photon antibunching and Cauchy–Schwarz share the same nonclassical properties in terms of the second-order correlation to determine the entanglement, as classified in Figure 2. The entanglement of optical phenomena happens when correlation is exhibited. This shows that entanglement is closely related to the autocorrelation of two system modes [26].

2.7. *Duan's criterion*

According to Duan's criterion [20], the two modes squeezing is related to the entanglement of the cavity field. Duan introduced the maximal entangled continuous variables which can be expressed as EPR-like operators [27],

$$u = |a|\, x_1 + \frac{1}{a} x_2, \tag{16}$$

$$v = |a|\, p_1 - \frac{1}{a} p_2. \tag{17}$$

The total variance must satisfy a lower boundary for an inseparable state

$$D = \triangle u^2 + \triangle v^2 < a^2 + \frac{1}{a^2}. \tag{18}$$

For a maximal entangled continuous variable, $a = 1$ and the RHS $\left| a^2 - 1/a^2 \right|$ reduces to zero. An experiment by Furusawa et al. [28] demonstrated that entanglement of a continuous variable was generated by two-mode squeezing which is also as described by Equation (7) where the coherent state occurs when identical quadrature variance is equal to $\frac{1}{2}$. Here, the steady state solution of the two-modes squeezing is defined as

$$\triangle c_\pm^2 = 1 + \langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle \pm 2\mathrm{Re}\left[\langle \hat{a}_1, \hat{a}_2 \rangle \right]. \tag{19}$$

Since

$$\triangle u^2 + \triangle v^2 = 2 \triangle c_-^2, \tag{20}$$

entanglement is achieved by the generation of the two-mode squeezing following the condition introduced by Duan [27]. Equation (19) is replaced into Equation (18) giving

$$\langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle < 2\mathrm{Re}\left[\langle \hat{a}_1, \hat{a}_2 \rangle \right]. \tag{21}$$

It is well known that the two-modes squeezing of cavity radiation is entangled if the quantum fluctuation satisfies the condition $\triangle u^2 + \triangle v^2 < 2$. This condition is important in the entanglement process because it leads to necessary inseparability criteria for continuous variables of the Gaussian state [27].

2.8. *Hillery–Zubairy criterion*

In another relevant study, Hillery and Zubairy introduced a two-mode state originated from the electromagnetic field where the operators were defined as [29]

$$L_1 = ab^\dagger + a^\dagger b, \tag{22}$$

$$L_2 = \mathrm{i}\left(ab^\dagger - a^\dagger b \right), \tag{23}$$

$$L_3 = a^\dagger a + b^\dagger b. \tag{24}$$

They employed two-mode squeezing and the Cauchy–Schwarz inequality to depict the entanglement where $SU(2)$ lie algebra, $J_i = L_i/2$ for $i = 1, 2, 3$ leads to satisfaction of the commutation relation as $[J_1, J_2] = \mathrm{i}J_3$. Then, the general equation for the uncertainty principle of variables must be satisfied

$$\left(\triangle L_1^2 \right) \left(\triangle L_2^2 \right) \geq \frac{|[J_1, J_2]|^2}{4}. \tag{25}$$

The total quadrature variance has the form of

$$(\triangle L_1)^2 + (\triangle L_2)^2 = 2\left[\langle (N_a+1)N_b\rangle + \langle N_a (N_b+1)\rangle - 2\left|\left\langle ab^\dagger\right\rangle\right|^2\right] \tag{26}$$

$$\simeq 2\left[\langle N_a+1\rangle \langle N_b\rangle + \langle N_a\rangle \langle N_b+1\rangle - 2\left|\langle a\rangle \left\langle b^\dagger\right\rangle\right|^2\right]. \tag{27}$$

The expectation values of two-mode are expressed as a product of a mode and b mode. The Schwarz inequality is applied where $|\langle a\rangle|^2 \leq \langle N_a\rangle$ and $|\langle b\rangle|^2 \leq \langle N_b\rangle$ [29], resulting in the product state in the form of

$$(\triangle L_1)^2 + (\triangle L_2)^2 \geq 2\left(\langle N_a\rangle + \langle N_b\rangle\right). \tag{28}$$

Equation (28) shows that the state is entangled if it satisfies the condition $\langle N_a N_b\rangle < \left|\left\langle ab^\dagger\right\rangle\right|^2$. This condition is employed in Hillery–Zubairy method using the relation of photon number in terms of annihilation and creation operators, $\hat{n} = \hat{a}^\dagger \hat{a}$ in two-mode entanglement to yield the inequality $\langle n_1\rangle \langle n_2\rangle < |\langle a_1 a_2\rangle|^2$ [20]. Furthermore, the inequality relation in the Hillery–Zubairy approach is applicable for entanglement of a multipartite system.

2.9. *Bell theorem*

Other than that, the Bell theorem is used to depict the entanglement according to a correlated state resulting with joint probability [21]. This theorem had been proved in an experiment using the Stern–Gerlach apparatus (SGA). The experiment involves anticorrelation of spin projection that resulted in different joint probability. The probability of half spin defined by the Bell theorem agreed with the SGA experiment in determining the orientation of the different angles, θ_a, θ_b and θ_c. The joint probabilities of ab, bc and ac are denoted as

$$P_{ab} = P\left(\alpha_{12}|\beta_{12}\right) + P\left(\alpha_{21}|\beta_{21}\right), \tag{29}$$

$$P_{bc} = P\left(\beta_{12}|\gamma_{12}\right) + P\left(\beta_{21}|\gamma_{21}\right), \tag{30}$$

$$P_{ac} = P\left(\alpha_{12}|\gamma_{12}\right) + P\left(\alpha_{21}|\gamma_{21}\right). \tag{31}$$

The total of two joint probabilities must be positive which indicates

$$P_{ab} + P_{bc} = P\left(\alpha_{12}|\gamma_{12}\right) + P\left(\alpha_{21}|\gamma_{21}\right) + P\left(\beta_{12}|\beta_{12}\right) + P\left(\beta_{21}|\beta_{21}\right) \tag{32}$$

$$= P_{ac} + P\left(\beta_{12}|\beta_{12}\right) + P\left(\beta_{21}|\beta_{21}\right). \tag{33}$$

The correlated state is said to be entangled when the sum of the joint probability between particles a and b with particles b and c is greater than the joint probability of particles a and c [21]:

$$P_{ab} + P_{bc} > P_{ac}. \tag{34}$$

2.10. *GHZ equality*

The well-known GHZ (Greenberger–Horne–Zeilinger) method determines the entanglement of the tripartite state, $|\psi\rangle_3 = 2^{-1/2}\left(|\uparrow_1\uparrow_2\uparrow_3\rangle - |\downarrow_1\downarrow_2\downarrow_3\rangle\right)$ [21]. According to their experiment, the entanglement state involves fair sampling of probability of the three states. Entanglement in tripartite state $|\psi\rangle_3$ is verified after including the eigenstate operators $\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}$, $\sigma_y^{(1)}\sigma_x^{(2)}\sigma_y^{(3)}$ and $\sigma_y^{(1)}\sigma_y^{(2)}\sigma_x^{(3)}$, where it represents

$$\sigma_x|\uparrow\rangle = |\downarrow\rangle, \sigma_x|\downarrow\rangle = |\uparrow\rangle, \sigma_y|\uparrow\rangle = i|\downarrow\rangle \text{ and } \sigma_y|\downarrow\rangle = -i|\uparrow\rangle.$$

As an example, the product of eigenstate and eigenstate operator can be defined as

$$\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}|\psi\rangle_3 = \sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}\frac{1}{2^{1/2}}\left(|\uparrow_1\uparrow_2\uparrow_3\rangle - |\downarrow_1\downarrow_2\downarrow_3\rangle\right) \tag{35}$$

$$= \frac{1}{2^{1/2}}\left[i^2|\uparrow_1\uparrow_2\uparrow_3\rangle - \left(i^2\right)|\downarrow_1\downarrow_2\downarrow_3\rangle\right] \tag{36}$$

$$= |\psi\rangle_3. \tag{37}$$

The product of eigenstate $|\psi\rangle_3$ and eigenstate operators $(\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)})$ results in eigenvalue $+1$. Similarly, the product of eigenstate, $|\psi\rangle_3$ with the other two eigenstate operators $(\sigma_y^{(1)}\sigma_x^{(2)}\sigma_y^{(3)})$ and $(\sigma_y^{(1)}\sigma_y^{(2)}\sigma_x^{(3)})$ also produces the same eigenvalue of $+1$. Therefore, the value of the eigenstate operator of $\sigma_x^{(3)}$ is $+1$ if $\sigma_y^{(1)}$ and $\sigma_y^{(1)}$ are equal to $+1$. However, if $\sigma_y^{(1)}$ and $\sigma_y^{(2)}$ are equal to $+1$ and -1 respectively, it will cause the value of $\sigma_x^{(3)}$ to become -1 [21]. In the entangled state case, the contradiction value of the eigenstate operator should be assigned the value -1 because the outcome of the hidden variable theory is predicted to be always $+1$.

$$\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}|\psi\rangle_3 = -|\psi\rangle. \tag{38}$$

The state is said to be entangled if and only if the product of the eigenstate $|\psi\rangle_3$ and eigenstate operators $\left(\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}\right)$ are equal to -1 [21].

2.11. *Negative Wigner function*

The Wigner function is a method used to calculate the classical probability distribution in terms of phase space and it can be implemented in the field of quantum mechanics. The general equation of the Wigner function can be obtained from statistical quantum theory. The Wigner function of a state $|\psi\rangle$ in terms of the density operator is described as [30]

$$W_\psi(x, p) \equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}\xi \exp\left(-\frac{\mathrm{i}p\xi}{\hbar}\right) \times \left\langle x + \frac{\xi}{2}\right|\rho\left|x - \frac{\xi}{2}\right\rangle. \tag{39}$$

The classical probability distribution requires Wigner function $W_{\psi}\,(x,\,p)$ to be positive. However, the Wigner function can be negative as a nonclassical probability distribution.

Our study employs the Wigner function to infer the presence of entanglement of a system due to delocalization through the nonclassical probability distribution. The Wigner function that exists for each quantum ensemble can be calculated from the density matrix equation. The double volume integration of the negative part of the Wigner function is expressed as

$$\delta\,(\psi) = \iint \left[\left|W_{\psi}(x,\,p)\right| - W_{\psi}(x,\,p)\right]\,\mathrm{d}x\,\mathrm{d}p \quad (40)$$

$$= \iint \left|W_{\psi}(x,\,p)\right|\,\mathrm{d}x\,\mathrm{d}p - 1, \quad (41)$$

where Wigner function $W_{\psi}(x,\,p)$ is defined as a function of position and momentum operator with normalization $\iint W_{\psi}(x,\,p)\,\mathrm{d}x\,\mathrm{d}p = 1$. When $\delta\,(\psi) = 0$, we have the coherent state. A squeezed vacuum state occurs for $\delta\,(\psi) < 0$ when the integral takes values in the region $0 \le \iint W_{\psi}(x,\,p)\,\mathrm{d}x\,\mathrm{d}p < 1$, resulting in entanglement.

2.12. *Logarithmic negativity*

Another relevant method is the logarithmic negativity which depicts the presence of entanglement for a two-mode state based on the negativity of the partial transposition [31]. The negative partial transpose must be parallel with respect to entanglement monotone in order to obtain the degree of entanglement. The logarithmic negativity is combined with negative partial transpose in another case where V represents the smallest eigenvalue of a simplistic matrix:

$$V = \left(\frac{\sigma + (\sigma^2 - 4\det\Upsilon)^{1/2}}{2}\right)^{1/2}, \quad (42)$$

where the invariant and covariance matrices are respectively denoted as

$$\sigma = \det A_1 + \det A_2 - 2\det A_{12}, \quad (43)$$

$$\Upsilon = \begin{pmatrix} A_1 & A_{12} \\ A_{12}^T & A_2 \end{pmatrix}. \quad (44)$$

For the initial cavity mode in a vacuum state, the covariant matrix is

$$\Upsilon = \begin{pmatrix} m & 0 & c & 0 \\ 0 & m & 0 & -c \\ c & 0 & n & 0 \\ 0 & -c & 0 & n \end{pmatrix}, \quad (45)$$

where $m = \langle \hat{a}_1^\dagger, \hat{a}_1\rangle + \langle \hat{a}_1, \hat{a}_1^\dagger\rangle$, $n = \langle \hat{a}_2^\dagger, \hat{a}_2\rangle + \langle \hat{a}_2, \hat{a}_2^\dagger\rangle$ and $c = \langle \hat{a}_1, \hat{a}_2\rangle + \langle \hat{a}_1^\dagger, \hat{a}_2^\dagger\rangle$. From the covariant matrix, we can see that m and n are symmetric [31]. Therefore, the pure state is symmetric and it fulfills the condition of $\pm c = (m^2 - 1)^{1/2}$. The correlations are determined by four local simplistic invariants

$$\det\Upsilon = \left(mn - c^2\right)\left(mn - (-c)^2\right), \quad (46)$$

$$\det A_1 = m^2, \quad (47)$$

$$\det A_2 = n^2, \quad (48)$$

$$\det A_{12} = c\,(-c). \quad (49)$$

The logarithmic negativity for a two-mode state is defined as $E_N = \max\left[0, -\log_2 V\right]$. The entanglement is achieved when E_N is positive within the region of the lowest eigenvalue of covariance matrix $V < 1$.

The entanglement of a quantum system is measurable at a certain range. Some approaches described earlier can be combined to establish a relation of inequalities which will strengthen the entanglement process. The density operator is an important element to determine the entanglement process. It is widely used for expressing the linear entropy which had been proven to be an effective method to quantify the entanglement [32]. The linear entropy is expressed as $L_A = 1 - \mathrm{Tr}_A\left(\rho_A^2\right)$ where the reduced density operator is $\rho_A = \mathrm{Tr}_B\,(\rho_{AB})$. The method is established according to time evolution to determine the bipartite entanglement process for the oscillator system. The oscillations in the system convert the pure state into a mixed state. All the methods described earlier can be used to study the entanglement between particles and radiation.

3. Quantifying quantum entanglement

The strength of entanglement can be quantified by measuring the degree of entanglement through concurrence and the entanglement of formation. Moreover, these two methods are related since concurrence provides an estimation for the formation of entanglement [33]. In quantum information of science, concurrence is an entanglement monotone defined for a mixed state of two qubits. For pure state $|\psi\rangle$ in the tensor product of Hilbert space $H_A \otimes H_B$, the concurrence is defined as

$$C\,(|\psi\rangle) = \left(2\left(1 - \mathrm{Tr}\left[\rho_A^2\right]\right)\right)^{1/2}, \quad (50)$$

where $\rho_A = \mathrm{Tr}_B\,[|\psi\rangle\,\langle\psi|]$. For a mixed state $\rho = \sum_i^N p_i\,|\psi_i\rangle\,\langle\psi_i|$ which has N statistical mixtures of pure states, the concurrence is extended to the convex roof

$$C\,(\rho) = \min_{\{p_i|\psi_i\rangle\}} \sum_i p_i C\,(|\psi_i\rangle), \quad (51)$$

where $p_i \ge 0$ and $\sum_i p_i = 1$. Concurrence is a measure of entanglement between any two systems that require minimal physical resource to prepare the quantum state. The quantity describes quantum phase transition in the interactions of a many-body system [34].

The formation of entanglement for pure state $|\psi\rangle = \sum_{ij} a_{ij}\,|ij\rangle \in H \otimes H$ is

$$E\,(|\psi\rangle) = -\mathrm{Tr}\left(\rho_A \log_2 \rho_A\right) = -\mathrm{Tr}\left(\rho_B \log_2 \rho_B\right), \quad (52)$$

where $\rho_B = Tr_A [|\psi\rangle \langle\psi|]$. For mixed state $\rho = \sum p_i |\psi_i\rangle \langle\psi_i|$, formation of entanglement is expressed as

$$E(\rho) = \min \sum p_i E(|\psi_i\rangle). \tag{53}$$

According to [35], the formation of entanglement is denoted as

$$E(\rho) = \varepsilon(C(\rho)), \tag{54}$$

where the function of entanglement $\varepsilon(C(\rho))$ is defined as

$$\varepsilon(C(\rho)) = h\left(\frac{1 + \left(1 - C(\rho)^2\right)^{1/2}}{2}\right) \tag{55}$$

with $h = -x \log_2 x - (1-x)\log_2(1-x)$ being the binary entropy function [36]. The function $E(\rho)$ monotonically increases following the range of $0 \le C(\rho) \le 1$ [37]. The important aspect of entanglement criteria based on concurrence is the existence of correlation between two subsystems which was discussed separately earlier. The strongest correlation resulted in the maximum value of the formation of entanglement since the possibility of concurrence is maximum here. Therefore, entanglement that corresponds to zero concurrence value can be defined by the expression of entanglement for any separable states [35]. The general expression for the formation of entanglement can also be constructed from the minimum value of a subensemble pure state,

$$E(\rho) = \min \sum_i p_i S\left(\rho_i^A\right). \tag{56}$$

The properties of entanglement were studied in [35], which emphasizes that the entanglement of subensemble state ρ_i cannot exceed the expected entanglement in the ρ state,

$$E(\rho) \ge \sum_i p_i E(\rho_i). \tag{57}$$

According to [38], the entanglement process poses a limitation when fixing the eigenvalues of ρ. Furthermore, the limitation arises when entanglement is shared among other states such as the bipartite state. If the states are in the n qubits system, the strength of the paired qubits increases to compete against the other pairs. The exact value of concurrence had been estimated based on lower bound and upper bound for the entanglement of formation [39]. The lower bound and upper bound correspond to the minimum value and maximum value of concurrence, respectively.

The entanglement of formation can be quantified for any $m \otimes n$ $(m \le n)$ when it satisfies the condition $\varepsilon(C(\rho)) \le E(\rho) \le \eta(C(\rho))$ considering that $\varepsilon(C)$ and $\eta(C)$ correspond to a maximum convex function and a minimum concave function respectively [39]. This condition proves that the concurrence monotonically increases within the limit.

4. Conclusion

We have extensively reviewed the concept of entanglement and described the different key quantities used to detect the presence of entanglement and even quantify it. We show that some of the criteria are mathematically related and therefore qualitatively correspond to each other. The criteria are classified by different systems involving discrete and continuous variables. The usefulness of each criterion depends on the properties and conditions of validity. For example, the usage of criteria depends on whether the criteria provide necessary or just sufficient condition and whether phase sensitivity is involved in the measurement of a system.

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