PHYS 690 Quantum and Nonlinear Optics

Problem set # 4 *(discussion date - November 15)*

P1 Using the expression for the second-order coherence function:
$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle^2}$,
show that for a number state $|n\rangle$ $g^{(2)}(0) = 1 - \frac{1}{n}$. Also calculate $g^{(2)}(0)$ for a coherent state.

P2 Calculate the output state after the 50/50 beam splitter when a two-photon state $|2\rangle$ is sent to one of its inputs.

P3 It is normally impossible to distinguish between coherent states $|\alpha\rangle$ and $|-\alpha\rangle$. There is, however, a measurement that can perform this task. Let's assume that one of these state (we don't know which) is inserted into the first port of a 50/50 beam splitter, and combined with a coherent state $|i\alpha\rangle$ in the other input port. Calculate the outputs and propose how to use the measurement results to tell which of two states it was. Will it always be possible to have a clear answer?

P4 Using Baker-Hausdorf lemma (below) one can prove that the unitary operator $\hat{U} = exp\left[i\frac{\pi}{4}(\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_0\hat{a}_1^\dagger)\right]$ describes the transformation between input and output modes of a 50/50 beam splitter. Moreover, it is easy to generalize it to the operator $\hat{U}_\theta = exp\left[i\frac{\theta}{2}(\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_0\hat{a}_1^\dagger)\right]$, which corresponds to the more general case where the beam splitter is *not* 50/50. Obtain the corresponding transformation of the mode operators and relate the angle θ with the transmission/reflection parameters r, t, r' and t'.
Baker-Hausdorf lemma:
$$e^{iG\lambda}Ae^{-iG\lambda} = A + i\lambda[G,A] + \frac{(i\lambda)^2}{2!}[G,[G,A]] + \ldots + \frac{(i\lambda)^n}{n!}\underbrace{[G,[G,[G,\ldots[G}_{n\ times},A]]]\ldots] + \ldots$$

P5 When dealing with measurements, it is often more convenient to talk about quadrature operators $\hat{X}_{1,2}$ rather than annihilation and creation operators. Calculate the transformation relations between the quadrature operators before and after a beam splitter.

P6 In class we discussed that optical losses degrade squeezing. To qualitatively estimate this effect it is common to model optical losses as a beam splitter. For example, a 10% loss is modeled as a 90/10 beam splitter, such that 90% of photons are transmitted, and ten per cent are reflected and "replaced" with coherent vacuum. Assume that the 5 dB squeezed vacuum field from the last week's problem falls into one input of such beam splitter, and the regular coherent vacuum - into the other. Calculate the amount of squeezing after the 10% losses (i.e. after the action of such fictional beam splitter).

P7 When discussing the homodyne detection in class, we have assume that the local oscillator is a classical field (i.e. has no fluctuations). Show that even if we use a "proper" coherent state to describe the local oscillator, the homodyne detector output is going to be the same as before, if the local oscillator field is strong enough.

P8 NOON states, i.e., the quantum superpositions of N photons in a form $\frac{1}{\sqrt{2}}(|N,0\rangle + |0,N\rangle)$ can be extremely beneficial for quantum phase measurements or quantum lithography. However, they are not easy to generate, especially for larger N. Suppose you want to produce such a state by sending N photons on a 50/50 beam splitter. What is the probability of success for such operation? Carry out the calculations for $N = 3$ and then show how the success probability scales with N.