PHYS 690 Quantum and Nonlinear Optics

Problem set # 1 *(discussion date - September 22)*

P1 Starting from the wave equation:
$-\nabla^2 \vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = -\mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$
derive the following equations of motion for the slowly-varying amplitude E_0 and phase ϕ , defined as $\vec{E} = \vec{e}_x E_0(z,t)\ e^{i[kz-\omega t+\phi(z,t)]}$:
$\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t} = -\frac{k}{2\epsilon_0}\mathrm{Im}(\mathcal{P}_0)$
$E_0\left(\frac{\partial \phi}{\partial z} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right) = \frac{k}{2\epsilon_0}\mathrm{Re}(\mathcal{P}_0)$

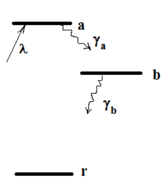
P2 Electron spin is a ubiquitous example of a two-level scheme, and thus the Block sphere is useful visualization for its quantum state. Let's assume that the electron is in a random superposition of "spin up" and "spin down" states $\psi = c_\uparrow|\uparrow\rangle + c_\downarrow|\downarrow\rangle$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of $\hat{S}_z$ operator.
(a) Using notation $\begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix} = \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}$ calculate average values of spin components $\hat{S}_z$, $\hat{S}_x$ and $\hat{S}_y$. Show the positions of the each operator's eigenfunctions on the Bloch sphere.
(b) Larmore precession The dynamical evolution of a spin-1/2 state can be viewed as the motion of a single point on the Bloch sphere. Describe the trajectory on the Bloch sphere of an arbitrary initial state, subject to the Hamiltonian $\hat{H}_B = \Omega_L \hat{S}_z$, corresponding the time evolution of an electron spin in the uniform magnetic field (Ω_L is known as a Larmor frequency, proportional to the applied magnetic field).

P3 In class we derived expression for Rabi oscillations in the resonance condition ($\Delta = 0$). Repeat the calculations for $\Delta \neq 0$, and find the value of the excited state population assuming the atom initially in the ground state $c_b(t=0) = 1$.

P4 Consider a two-level atom interacting with a resonant electromagnetic field ($\Delta = 0$) and initially in its ground state. Assume that a Rabi frequency $\Omega(t)$ is constant for a time τ and zero otherwise. Under what condition(s) will the system evolve to (a) the state $|\psi\rangle = (|a\rangle + |b\rangle)/\sqrt{2}$ and (b) the state $|\psi\rangle = |a\rangle$. A pulse that permits to achieve the first state is called $\pi/2$ pulse, and a pulse that permits to reach the second state is called a π-pulse. Can you guess why?

P5 In class we consider one electromagnetic field interacting with a two-level system. Now consider a three-level system, in which some of the levels are coupled by different electromagnetic fields (each fields interacts drives only one atomic transition). The equation below describe the time evolution of the wave function coefficients $|\psi\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$:
$\dot{c}_1 = \frac{i}{2}\mathcal{R}c_2$
$\dot{c}_2 = \frac{i}{2}\mathcal{R}(c_1 + c_3)$
$\dot{c}_3 = \frac{i}{2}\mathcal{R}c_2$ From the form of these equations deduce the interaction Hamiltonian and the values of Rabi frequencies for all participating fields.

P6 Consider an ensemble of two-level atoms, 30% of which are in the state $(|a\rangle + |b\rangle)/\sqrt{2}$, 50% are in the state $(|a\rangle - 3|b\rangle)/\sqrt{10}$, and 20% are in the state $|b\rangle$. Find the density matrix $\hat{\rho}$ of this system, and determine the probability for the atoms to be in the ground state $|b\rangle$



P7 Consider a three-level system shown. The reservoir level r is the ground-state. Upper level a decays into level b with the decay rate γ_a, and the lower level b decays into the ground state with the rate γ_b. An electric discharge effectively repopulates the state a at a rate λ. Write down the equations for populations of the levels a and b, and find the relationship between $\lambda, \gamma_a, \gamma_b$ that produces a steady-state population inversion.