Homework #1 (solution)

Problem 1:

Slowly varying approximation $\left|\frac{\partial E_0}{\partial t}\right| \ll \omega E_0$; $\left|\frac{\partial \varphi}{\partial t}\right| \ll \omega$

$\left|\frac{\partial E_0}{\partial z}\right| \ll k E_0$; $\left|\frac{\partial \varphi}{\partial t}\right| \ll k$

$\left|\frac{\partial \mathcal{P}}{\partial t}\right| \ll \omega \mathcal{P}$; $\left|\frac{\partial \mathcal{P}}{\partial z}\right| \ll k \mathcal{P}$

We also assume that the effect of the medium is weak, so that $\mathcal{P}/E_0 \ll 1$

$$\frac{\partial E}{\partial t} = -i\omega E + \underbrace{\frac{\partial E_0}{\partial t} e^{ikz-i\omega t+i\varphi} + i\frac{\partial \varphi}{\partial t} E}_{\text{small terms}}$$

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2 E - 2i\omega\left(\frac{\partial E_0}{\partial t} e^{ikz-i\omega t+i\varphi} + i\frac{\partial \varphi}{\partial t} E\right) + \left\{\text{terms } \frac{\partial^2 E_0}{\partial t^2}, \frac{\partial^2 \varphi}{\partial t^2}, \left(\frac{\partial E_0}{\partial t}\frac{\partial \varphi}{\partial t}\right)\right\}$$

neglect due to smallness

Similarly

$$\frac{\partial^2 E}{\partial z^2} \simeq -k^2 E + 2ik\left(\frac{\partial E_0}{\partial t} e^{ikz-i\omega t+i\varphi} + i\frac{\partial \varphi}{\partial t} E\right)$$

And we are keeping only the leading term for P

$$\frac{\partial^2 P}{\partial t^2} = -\omega^2 \mathcal{P} e^{ikz-i\omega t+i\varphi}$$

Wave equation

$$-\frac{\partial^2 E}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = -\mu_0 \frac{\partial^2 P}{\partial t^2}$$

$$k^2 E - 2ik\left(\frac{\partial E_0}{\partial z} + i\frac{\partial \varphi}{\partial z} \cdot E_0\right)\cancel{e^{ikz-i\omega t+i\varphi}} - \frac{\omega^2}{c^2}E - \frac{2i\omega}{c^2}\left(\frac{\partial E_0}{\partial t} + i\frac{\partial \varphi}{\partial t}E_0\right)\cancel{e^{ikz-i\omega t+i\varphi}} =$$

$$= \mu_0 \omega^2 \mathcal{P}\cancel{e^{ikz-i\omega t+i\varphi}}$$

$$\left(k^2 - \omega^2/c^2\right)E = 0 \Rightarrow k = \omega/c$$

$$-2ik\left[\frac{\partial E_0}{\partial z} + iE_0\frac{\partial \varphi}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t} + iE_0\frac{\partial \varphi}{\partial t}\right] = \mu_0\omega^2 \mathcal{P}$$

$$\left[\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t}\right] + iE_0\left(\frac{\partial \varphi}{\partial z} + \frac{1}{c}\frac{\partial \varphi}{\partial t}\right)\Big] = +i\frac{\mu_0\omega^2}{2k}\mathcal{P}$$

$$\boxed{\frac{\mu_0\omega^2}{2k} = \frac{\mu_0\omega c}{2} = \frac{k}{2\varepsilon_0} \quad \text{since } c^2 = 1/\mu_0\varepsilon_0}$$

Separating real and imaginary parts

$$\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t} = -\frac{k}{2\varepsilon_0}\,\mathrm{Im}(\mathcal{P})$$

$$\frac{\partial \varphi}{\partial z} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = \frac{k}{2\varepsilon_0}\,\mathrm{Re}(\mathcal{P})$$

2) (a) $|\psi\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\theta/2 \\ e^{i\varphi/2}\sin\theta/2 \end{pmatrix}$

$$\langle \hat{S}_x\rangle = \langle\psi|\hat{S}_x|\psi\rangle = \langle\psi|\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}|\psi\rangle =$$

$$= \left(e^{i\varphi/2}\cos\frac{\theta}{2} \quad e^{-i\varphi/2}\sin\frac{\theta}{2}\right)\begin{pmatrix} e^{i\varphi/2}\sin\theta/2 \\ e^{-i\varphi/2}\cos\theta/2\end{pmatrix} =$$

$$= (e^{i\varphi}+e^{-i\varphi})\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \cos\varphi\sin\theta$$

Similarly

$$\langle\hat{S}_y\rangle = \langle\psi|\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}|\psi\rangle = i(-e^{i\varphi}+e^{-i\varphi})\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \sin\varphi\sin\theta$$

$$\langle\hat{S}_z\rangle = \langle\psi|\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}|\psi\rangle = \cos^2\theta/2 - \sin^2\theta/2 = \cos\theta$$

It is easy to notice that $\langle\hat{S}_x\rangle$, $\langle\hat{S}_y\rangle$ and $\langle\hat{S}_z\rangle$ corresponds to the x, y and z component of a unit vector in sperical coordinates

(b) In the presence of magnetic field the $|\uparrow\rangle$ and $|\downarrow\rangle$ states ~~are~~ become non-degenerate, shifting by $\pm\hbar\Omega_L/2$

Thus, the time evolution of the original quantum state is

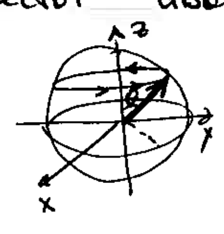
$$|\psi(t)\rangle = \begin{pmatrix} e^{-i\varphi/2+i\Omega_L t/2}\cos\frac{\theta}{2} \\ e^{i\varphi/2-i\Omega_L t/2}\sin\frac{\theta}{2}\end{pmatrix} = \begin{pmatrix} e^{-i\varphi(t)/2}\cos\theta/2 \\ e^{i\varphi(t)/2}\sin\theta/2\end{pmatrix}$$

$\boxed{\text{where } \varphi(t) = \varphi - \Omega_L t}$

Thus, using results from (a)

$\langle\hat{S}_x(t)\rangle = \cos(\varphi-\Omega_L t)\sin\theta$, $\langle\hat{S}_y(t)\rangle = \sin(\varphi-\Omega_L t)\sin\theta$, $\langle\hat{S}_z\rangle = \cos\theta$

That corresponds to the rotation of the unit vector about z-axis at frequency Ω_L



HW # 2

Homework #2 (solutions)

P1.

$$\begin{cases} \dot{c}_1 = \frac{i}{2} R c_2 \\ \dot{c}_2 = \frac{i}{2} R (c_1 + c_3) \\ \dot{c}_3 = \frac{i}{2} R c_2 \end{cases}$$

$$\because \quad \dot{c}_\ell (t) = -\frac{i}{\hbar} \sum_k c_k (t) e^{i\omega_{\ell k} t} \langle \ell | H_I | k \rangle$$

$$\langle 1 | H | 1 \rangle = \langle 2 | H | 2 \rangle = \langle 3 | H | 3 \rangle = 0 .$$

$$\therefore \quad \dot{c}_1 = -\frac{i}{\hbar} c_2 e^{i\omega_{12} t} \langle 1 | H | 2 \rangle \underbrace{- \frac{i}{\hbar} c_3 e^{i\omega_{13} t} \langle 1 | H | 3 \rangle}_{=0}$$

$$\therefore \quad \langle 1 | H | 3 \rangle = 0 .$$

$$-\frac{i}{\hbar} e^{i\omega_{12} t} \langle 1 | H | 2 \rangle = \frac{i}{2} R .$$

$$\langle 1 | H | 2 \rangle = \frac{-\hbar R}{2} e^{i\omega_{12} t}$$

$$\dot{c}_3 = \underbrace{-\frac{i}{\hbar} c_1 e^{i\omega_{31} t} \langle 3 | H | 1 \rangle}_{=0} - \frac{i}{\hbar} c_2 e^{i\omega_{32} t} \langle 3 | H | 2 \rangle$$

$$\langle 3 | H | 1 \rangle = 0$$

$$-\frac{i}{\hbar} e^{i\omega_{32} t} \langle 3 | H | 2 \rangle = \frac{i}{2} R$$

$$\langle 3 | H | 2 \rangle = \frac{-\hbar R}{2} e^{i\omega_{32} t} .$$

$$\dot{c}_2 = -\frac{i}{\hbar} c_1 e^{i\omega_{21} t} \langle 2|H|1\rangle - \frac{i}{\hbar} c_3 e^{i\omega_{23} t} \langle 2|H|3\rangle .$$

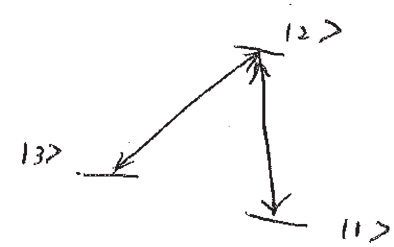
$$\begin{cases} -\frac{i}{\hbar} e^{i\omega_{21} t} \langle 2|H|1\rangle = \frac{i}{2} R \\ -\frac{i}{\hbar} e^{i\omega_{23} t} \langle 2|H|3\rangle = \frac{i}{2} R. \end{cases}$$

$$\langle 2|H|1\rangle = \frac{-\hbar R}{2} e^{-i\omega_{21} t}$$

$$\langle 2|H|3\rangle = -\frac{\hbar R}{2} e^{-i\omega_{23} t} .$$

$$\therefore \quad H_I = -\frac{\hbar R}{2} \begin{pmatrix} 0 & e^{i\omega_{12} t} & 0 \\ e^{i\omega_{21} t} & 0 & e^{i\omega_{23} t} \\ 0 & e^{i\omega_{32} t} & 0 \end{pmatrix}$$

$$\omega_{\ell k} = \frac{E_\ell - E_k}{\hbar} .$$



Rabi oscillations for $\Delta \neq 0$

$$\begin{cases} \dot{c}_a = i\Delta c_a + i\Omega c_b \\ \dot{c}_b = i\Omega^* c_a \end{cases}$$

$$\ddot{c}_a = i\Delta \dot{c}_a + i\Omega \dot{c}_b = i\Delta \dot{c}_a - |\Omega|^2 c_a$$

$$c_a = e^{i\lambda t} : \quad -\lambda^2 = -\lambda\Delta - |\Omega|^2$$

$$\lambda^2 - \lambda\Delta - |\Omega|^2 = 0$$

$$\lambda_{1,2} = \frac{\Delta}{2} \pm \sqrt{\frac{\Delta^2}{4} + |\Omega|^2}$$

$$c_a(t) = A_1 \cdot e^{i\lambda_1 t} + A_2 e^{i\lambda_2 t} = e^{i\frac{\Delta t}{2}} \left[A_1 e^{i\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}\,t} + \right.$$

$$\left. + A_2 e^{-i\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}\,t} \right]$$

$$c_a(0) = 0 \quad \Rightarrow \quad A_1 = -A_2 = A/2i$$

$$c_a(t) = A e^{i\Delta t/2} \sin\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}\cdot t$$

$$c_b(t) = \frac{1}{i\Omega}(\dot{c}_a - i\Delta c_a) = \frac{1}{i\Omega} A \left[\frac{i\Delta}{2} c_a + \sqrt{\frac{\Delta^2}{4}+|\Omega|^2}\, e^{i\frac{\Delta t}{2}} \cos\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}\,t \right.$$

$$\left. - i\Delta c_a \right] = \frac{A}{i\Omega} e^{i\frac{\Delta t}{2}} \left[\sqrt{\frac{\Delta^2}{4}+|\Omega|^2} \cos\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}\,t - \frac{\Delta}{2} \sin\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}\,t \right.$$

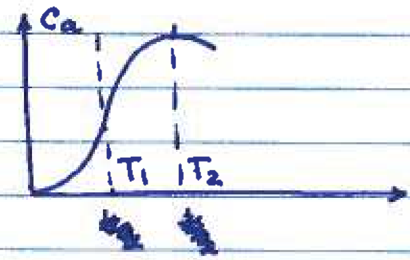
since $\quad c_b(t=0) = 1 = \dfrac{A}{i\Omega}\sqrt{\dfrac{\Delta^2}{4}+|\Omega|^2}$

$$A = \frac{i\Omega}{\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}}$$

$$P_a(t) = |c_a|^2 = \frac{\Omega^2}{\frac{\Delta^2}{4}+|\Omega|^2} \sin^2\left(\sqrt{\frac{\Delta^2}{4}+|\Omega|^2}\cdot t\right)$$

Rabi oscillations

$$|\psi\rangle = \cos|\Omega|t\,|b\rangle + i\frac{\Omega}{|\Omega|}\sin|\Omega|t\,|a\rangle$$



a) $|\psi\rangle_{target} = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$

$T_1|\Omega| = \frac{\pi}{4}$ $\Rightarrow$ $T_1 = \frac{\pi}{4|\Omega|}$

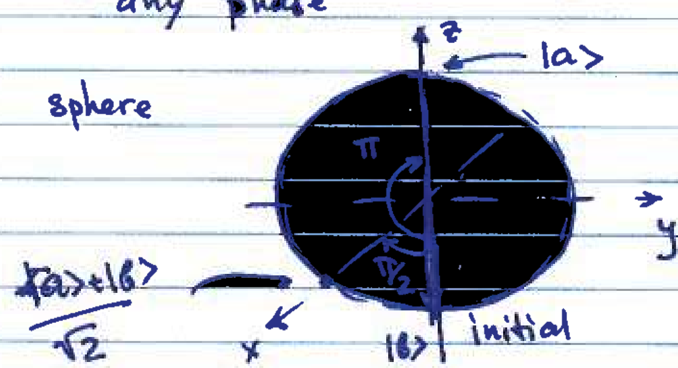
The phase of the optical field must be $-\pi/2$, so that

$$\frac{\Omega}{|\Omega|} = e^{i\varphi} = -i$$

b) $|\psi\rangle_{target} = |a\rangle$ $\quad T_2|\Omega| = \frac{\pi}{2} \quad T_2 = \frac{\pi}{2|\Omega|}$

any phase

Bloch sphere



5)

$$\begin{cases}\dot{c}_1 = \frac{i}{2}Rc_2\\ \dot{c}_2 = \frac{i}{2}Rc_1 + \frac{i}{2}Rc_3\\ \dot{c}_3 = \frac{i}{2}Rc_2\end{cases}$$

Since there is no terms $\dot{c}_i = i\Delta c_i + \ldots$ we can conclude that there is no detunings (i.e. all laser fields are tuned exactly on resonances)

Then, using $i\hbar\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle$

$$c_i = -\frac{i}{\hbar}\sum_{k\neq i} c_k\langle i|\hat{H}|k\rangle$$

$$c_1 = -\frac{i}{\hbar}\left(c_2\langle 1|\hat{H}|2\rangle + c_3\langle 1|\hat{H}|3\rangle\right)$$

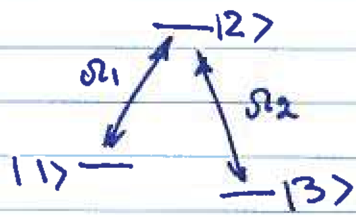
comparing ~~the~~ to the equations: $\langle 1|\hat{H}|3\rangle = 0$

$$-i/\hbar\,\langle 1|\hat{H}|2\rangle = \frac{i}{2}R$$

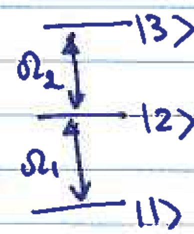
similarly $-i/\hbar\,\langle 3|\hat{H}|2\rangle = \frac{i}{2}R$

$$\langle 1|\hat{H}|2\rangle = -\wp_{12}\Omega_1 \Rightarrow \Omega_1 = R/2$$

$$\langle 3|\hat{H}|2\rangle = -\wp_{23}\Omega_2 \Rightarrow \Omega_2 = R/2$$



or



6. $|\psi_1\rangle = \dfrac{|a\rangle + |b\rangle}{\sqrt{2}} \Rightarrow |\psi_1\rangle\langle\psi_1| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

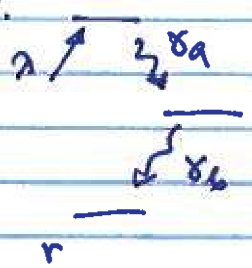
$|\psi_2\rangle = \dfrac{|a\rangle - 3|b\rangle}{\sqrt{10}} \Rightarrow |\psi_2\rangle\langle\psi_2| = \begin{pmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{pmatrix}$

$|\psi_3\rangle = |b\rangle \qquad |\psi_3\rangle\langle\psi_3| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\hat{\rho} = \sum p_i |\psi_i\rangle\langle\psi_i| = 0.3 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + 0.5 \begin{pmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{pmatrix} +$$

$$+ 0.2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/5 & 0 \\ 0 & 4/5 \end{pmatrix}$$

7.



$\dot{\rho}_{aa} = \lambda - \gamma_a \rho_{aa}$

$\dot{\rho}_{bb} = \gamma_a \rho_{aa} - \gamma_b \rho_{bb}$

Steady-state solution

$\rho_{aa} = \dfrac{\lambda}{\gamma_a} \qquad \rho_{bb} = \dfrac{\lambda}{\gamma_b}$

$$\rho_{aa} - \rho_{bb} = \lambda\left(\frac{1}{\gamma_a} - \frac{1}{\gamma_b}\right)$$

Population inversion $\rightarrow$ $\gamma_a < \gamma_b$ for any λ.

[redacted]