Problem 1 (20 points)

a) (5 points) Using the generation function for Legendre polynomials, prove the identity:

$$\sum_{m=0}^{\infty} P_m(\cos\theta) = \frac{1}{2\sin(\theta/2)}$$

b) (5 points) Decompose x^2 to the sum of the Legendre polynomials.

c) (10 points) Using results from a) and b) show that

$$\int_0^{\pi} \frac{\sin\theta\cos^2\theta}{\sin(\theta/2)} d\theta = \frac{28}{15}$$

a) Generation function

$$\Phi(h, x=\cos\theta) = \sum_{m=0}^{\infty} P_m(\cos\theta)\, h^m = \frac{1}{\sqrt{1+h^2-2h\cos\theta}}$$

$h = 1$

$$\sum_{m=0}^{\infty} P_m(\cos\theta) = \frac{1}{\sqrt{2-2\cos\theta}} = \frac{1}{2\sin\theta/2}$$

b) $P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$$x^2 = \frac{2}{3}\left[\frac{3}{2}x^2 - \frac{1}{2}\right] + \frac{1}{3} = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

c) $$\int_0^{\pi} \frac{\sin\theta\cos^2\theta}{\sin(\theta/2)} d\theta = 2\int_0^{\pi} \sin\theta\cos^2\theta \cdot \left[\sum_{m=0}^{\infty} P_m(\cos\theta)\right] d\theta = 2\int_{-1}^{1} x^2 \sum_{m=0}^{\infty} P_m(x)\, dx =$$

$$= 2\int_{-1}^{1} \left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] \sum_{m=0}^{\infty} P_m(x)\, dx = \{\text{due to orthogonality}\} =$$

$$= \frac{4}{3}\int_{-1}^{1} P_2^2(x)\,dx + \frac{2}{3}\int_{-1}^{1} P_0^2(x)\,dx = \frac{4}{3}\cdot\frac{2}{5} + \frac{2}{3}\cdot 2 = \frac{28}{15}$$

Problem 2 (35 points)

a) (15 points) Using the generation function $\Phi(x,t) = e^{-t^2+tx} = \sum_{n=0}^{\infty} X_n(x)\frac{t^n}{n!}$, derive two recurrence relations for polynomials X_n.

b) (15 points) Using the recurrence relations, show that the differential equation for X_n looks like $2y''-xy'+ny=0$.

c) (5 points) The polynomials $X_n(x)$ are very closely related to one of the "famous" polynomials we discussed in class. What is this relation?

a) $\Phi(x,t) = e^{-t^2+tx} = \sum_{n=0}^{\infty} X_n(x)\frac{t^n}{n!}$

$\frac{\partial}{\partial x}\Phi(x,t) = te^{-t^2+tx} = t\cdot\Phi = \sum_{n=0}^{\infty} X_n'(x)\frac{t^n}{n!}$

$\sum_{n=0}^{\infty} X_n \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} X_n'(x)\frac{t^n}{n!} \Rightarrow \boxed{X_n' = nX_{n-1}}$ $(*)$

$\frac{\partial}{\partial t}\Phi(x,t) = (-2t+x)e^{-t^2+tx} = (-2t+x)\Phi = \sum_{n=1}^{\infty} X_n \frac{t^{n-1}}{(n-1)!}$

$-2\sum_{n=0}^{\infty} X_n \frac{t^{n+1}}{n!} + \sum_{n=0}^{\infty} xX_n\frac{t^n}{n!} = \sum_{n=1}^{\infty} X_n\frac{t^{n-1}}{(n-1)!}$

$\boxed{-2nX_{n-1} + xX_n = X_{n+1}}$ $(**)$

b) From $(*)$ $nX_{n-1} = X_n'$

$-2X_n' + xX_n = X_{n+1}$

Differentiate the equation once

$-2X_n'' + xX_n' + X_n = X_{n+1}'$

From $(*)$ $(n+1)X_n = X_{n+1}'$

$-2X_n'' + xX_n' + X_n = nX_n + X_n$

$2X_n'' - xX_n' + nX_n = 0$

c) Comparing $e^{-t^2+tx} = \sum_{n=0}^{\infty} X_n(x)\frac{t^n}{n!}$

and $e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$

it is clear that $\underline{X_n(2x) = H_n(x)}$

Problem 3 (15 points)

Show that: $\int_0^1 (1-\rho^2) J_0(\rho)\rho \, d\rho = 2J_2(1)$

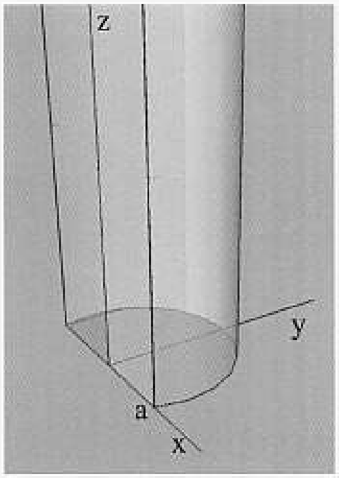
$$\int_0^1 (1-\rho^2) J_0(\rho)\rho \, d\rho = \int_0^1 \rho J_0(\rho)\, d\rho - \int_0^1 \rho^3 J_0(\rho)\, d\rho =$$

$$= \int_0^1 \frac{d}{d\rho}\left[\rho J_1(\rho)\right] d\rho - \int_0^1 \rho^2 \frac{d}{d\rho}\left[\rho J_1(\rho)\right] d\rho =$$

$$= \underbrace{\rho J_1(\rho)\Big|_0^1}_{= J_1(1)} - \underbrace{\rho^3 J_1(\rho)\Big|_0^1}_{= J_1(1)} + 2\int_0^1 \rho^2 J_1(\rho)\, d\rho = 2\int_0^1 \frac{d}{d\rho}\left[\rho^2 J_2(\rho)\right] d\rho =$$

$$= 2\rho^2 J_2(\rho)\Big|_0^1 = 2J_2(1)$$

Problem 4 (30 points)



a) (20 points) The Laplace equation in cylindrical coordinates is the following:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r}\frac{\partial^2 u}{\partial \phi^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

Separate the variables, and then write down ***the general expression*** for the temperature distribution in a semi-infinite half-cylinder (shown in the picture), if all its vertical sides are maintained at T=0, with non-zero temperature in the bottom. *Hint*: your answer will be a series of products of functions of r, ϕ and z – with some coefficient(s).

b) (10 points) Assuming that the temperature distribution on the bottom is $T_0(r,\phi)=f(r)g(\phi)$, write down the expression(s) for the coefficient in the solution series in terms of functions f and g.

a) $u = R(\rho)\,\Phi(\varphi)\,Z(z)$

$$\frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{1}{r^2}\underbrace{\frac{\Phi''}{\Phi}}_{-m^2}+\underbrace{\frac{Z''}{Z}}_{k^2}=0$$

since the solution for Φ is oscillatory, and do Z is exponentially decaying

$Z(z) = e^{-k^2 z^2}$, $\Phi(\varphi) = \left\{\begin{matrix}\cos m\varphi \\ \sin m\varphi\end{matrix}\right\}$ where m is integer

The equation for R is

$r\frac{d}{dr}(rR') - m^2R + k^2r^2R = 0$ or

$r^2R'' + rR' + (k^2r^2 - m^2) = 0$ Bessel equation

$R(r) = J_m(kr)$

Boundary conditions: $u(r=a,\varphi,z)=0 \Rightarrow R(r) = J_m(\alpha_i^{(m)}\, r/a)$ $\quad k = \alpha_i^{(m)}/a$

where $\alpha_i^{(m)}$ is the i-th zero of $J_m(x)$

$u(r,\varphi=0,z) = u(r,\varphi=\pi,z) = 0 \Rightarrow \Phi_m(\varphi) = \sin m\varphi$

So the general solution is

$$u(r,\varphi,z) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} A_{im}\, J_m(\alpha_i^{(m)}\, r/a)\,\sin m\varphi\; e^{-\alpha_i^{(m)2} z^2/a^2}$$

b) Bottom $u(r,\varphi,z=0) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} A_{im}\, J_m(\alpha_i^{(m)} r/a)\sin m\varphi = f(r)\,g(\varphi)$

First we should decompose $g(\varphi)$ into a sine Fourier series

$g(\varphi) = \sum_{m=0}^{\infty} G_m \sin m\varphi$, where $G_m = \frac{1}{\pi}\int_0^{\pi} g(\varphi)\sin m\varphi\, d\varphi$

Thus

$$\sum_{i=1}^{\infty} A_{im} J_m(\alpha_i^{(m)} r/a) = G_m f(r)$$

Using the orthogonality of the Bessel functions:

$$\sum_{i=1}^{\infty} A_{im} \underbrace{\int_0^a J_m(\alpha_i^{(m)} r/a) J_m(\alpha_j^{(m)} r/a)\, r\,dr}_{\frac{1}{2} J_{m+1}^2(\alpha_j^{(m)}) a^2 \delta_{ij}} = G_m \int_0^a r f(r) J_m(\alpha_j^{(m)} r/a)\, dr$$

$$\boxed{A_{im} = \frac{2 G_m}{J_{m+1}^2(\alpha_j^{(m)}) a^2} \int_0^a r f(r) J_m(\alpha_j^{(m)} r/a)\, dr}$$