Lecture #4
2009

Five basic Taylor series [these are easy to calculate analytically]

$$e^x = 1 + x + \frac{x^2}{2} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{6} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{n}{p} x^n \quad \text{where} \quad \binom{n}{p} = \frac{p(p-1)\dots(p-n)}{n!}$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x + \dots$$

Properties of the power series.

1. The power series is unique for each function.
2. Two power series can be added, subtracted, and multiplied; the resulting series will converge at least in the common interval of convergence.
3. Power series can be integrated and differentiated as many times as desired within their convergence interval.
4. One power series can be substituted into another.
5. Two power series can be divided, but the convergence interval may be smaller than that of the original series.

Example 1: addition of series [check $e^{i\varphi} = \cos\varphi + i \sin\varphi$]

$$\cos\varphi + i\sin\varphi = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \underbrace{\sum_{n=0}^{\infty} i^{2n} \frac{x^{2n}}{(2n)!}}_{\text{even terms of } \sum_{k=0}^{\infty} \frac{(ix)^k}{k!}} + \underbrace{i \sum_{n=0}^{\infty} i^{2n} \frac{x^{2n+1}}{(2n+1)!}}_{\text{odd terms of } \sum_{k=0}^{\infty} \frac{(ix)^k}{k!}}$$

$$= \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = e^{ix}$$

Example 2: multiplication of two series [check $\cos x \sin x = \frac{1}{2}\sin 2x$]

$$\sin x \cdot \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \cdot \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} =$$

a) (if we need only a few first terms)

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\right) = x + \left[-\frac{x^3}{2} - \frac{x^3}{6}\right] + \left[\frac{x^5}{120} + \frac{x^5}{12} + \frac{x^5}{24}\right] - \ldots$$

$$= x - \frac{2}{3}x^3 + \frac{16}{120}x^5 - \ldots = \frac{1}{2}\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots\right] = \frac{1}{2}\sin 2x$$

Example 3: division of series

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - x^3/6 + x^5/120 - \ldots}{1 - x^2/2 + x^4/24 - \ldots}$$

$\tan x$ is an <u>odd</u> function so its series doesn't contain x^{2n}

Approach 1: $\tan x = a_1 x + a_3 x^3 + a_5 x^5 + \ldots$

$$\tan x \cdot \cos x = \sin x \; ; \quad (a_1 x + a_3 x^3 + a_5 x^5 + \ldots)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \right) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

x: $a_1 = 1$

x^3: $a_3 - \frac{a_1}{2} = -\frac{1}{6}$ $\qquad a_3 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

x^5: $a_5 - \frac{1}{2}a_3 + \frac{1}{24}a_1 = \frac{1}{120}$ $\qquad a_5 = \frac{1}{120} - \frac{1}{24} + \frac{1}{6} = \frac{2}{15}$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

There exists a general expression for $\tan x$

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_n}{(2n)!} x^{2n-1}$$

where B_n are Bernulli numbers

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$$

Approach 2: $$\tan x = \frac{x - x^3/6 + x^5/120}{1[- x^2/2 + x^4/24 -]} = \frac{x - x^3/6 + x^5/120}{1 - \underbrace{[x^2/2 - x^4/24 +]}_{=y}} =$$

$\left[\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n = 1 + y + y^2\right]$ geometrical progression

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\left(1 + \left[\frac{x^2}{2} - \frac{x^4}{24}\right] + \left[\frac{x^2}{2} - \frac{x^4}{24}\right]^2 + \right) =$$

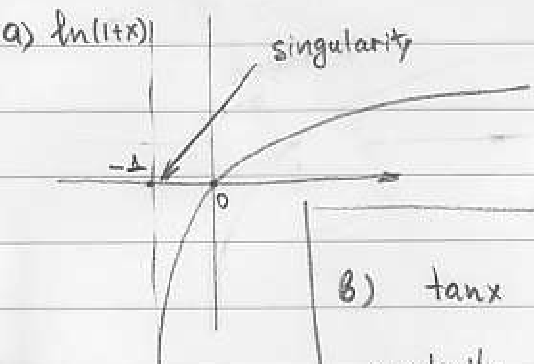
$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\left(1 + \frac{x^2}{2} + \overbrace{\left(\frac{1}{4} - \frac{1}{24}\right)}^{\frac{5}{24}} x^4 + \ldots\right) = x + \frac{x^3}{3} + \frac{2x^5}{15} +$$

Such substitutions may be very useful!
Suppose you need to estimate the value of $\frac{x^3}{1-\sqrt{1-x^2}}$ for small x

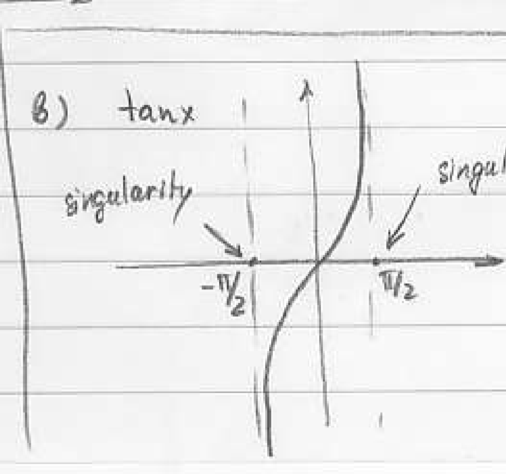
$$\frac{x^3}{1-\sqrt{1-x^2}} \approx \frac{x^3}{1-\left(1-\frac{x^2}{2}-\frac{x^4}{8}\right)} = \frac{x^3}{\frac{x^2}{2}+\frac{x^4}{8}} = \frac{2x}{1+\frac{x^2}{4}} \approx 2x\left(1-\frac{x^2}{4}\right) = 2x - \frac{x^3}{2}$$

Interval of convergence
If the analytical expression for the Taylor series is known we can use known convergence tests.
[Check that e^x, $\cos x$, $\sin x$ converge for any x, $\ln x$ and $(1+x)^p$ converge for $|x|<1$ using the ratio test]
However, often the expression for a_n is not known, so we have to use function itself to extract the information about its Taylor series convergence

To determine <u>the interval of convergence</u> we have to check in what region of the complex plane the function is <u>analytic</u>. The Taylor series will converge inside the circle around zero (or z_0 for $\Sigma a_n(z-z_0)^n$ series) that extends to the nearest singularity point.

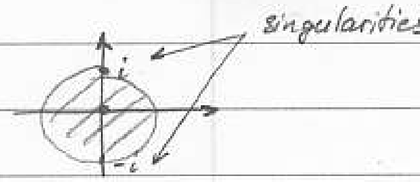


thus $\ln(1+x)$ Taylor series will converge inside $|z|<1$ circle. [or for $|x|<1$ in real axis]



Taylor series for $\tan x$ converges for $|x| < \pi/2$

One have to be very careful if the singularity is on imaginary axis!



$\frac{1}{1+x^2} \Rightarrow 1+z^2=0 \quad z=\pm i$

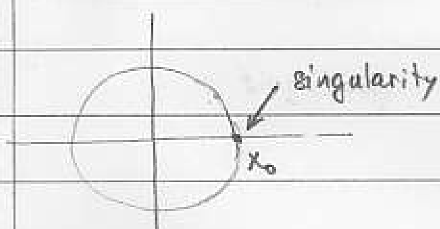
series converges for $|z|<1$

or (in real axis $|x|<1$)

For example, if one needs to find the convergence interval for $\frac{1}{1+\sin^2 x} \Rightarrow$ singularity at $1+\sin^2 z = 0$

$\underline{\sin z = \pm i}$

Lauren series.



We can construct a Taylor series for $|x|<x_0$ $\quad f(x) = \sum_{n=0}^{\infty} a_n x^n$

For $|x|>x_0$ one has to construct a Lauren series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n \frac{1}{x^n}$$

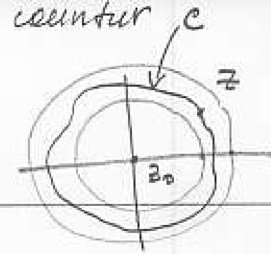
A Lauren series can have finite or infinite number of b_n terms.

Examples: a) $f(x) = \frac{\sin x}{x^2} = \frac{1}{x^2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \underline{\underline{\frac{1}{x}}} + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n+1)!}$

b) $\frac{1}{1+x^2} \overset{(x>1)}{=} \frac{1}{x^2} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{x^2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{x^{2n}} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{x^{2n+2}}$

How do we know what series to construct?

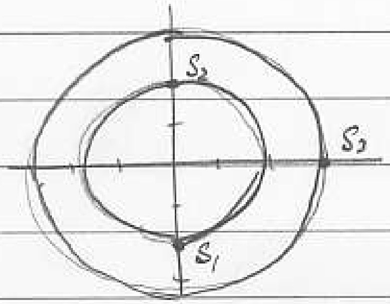
If the function is analytic between two circles centered at $z=0$ (or $z=z_0$), then $f(z)$ can be expanded in a unique Laurent series in this region.



$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad b_n = \frac{1}{2\pi i} \oint_C f(z)(z-z_0)^{n-1} dz$$

Example $\quad f(x) = \dfrac{1}{(x^2+4)(x-3)}$



The circles drawn throw three singularities $S_{1,2} = \pm 2i$ and $S_3 = 3$ (centered at $z=0$) divides the space into 3 regions. For $|z|<2$ $f(z)$ can be expanded into a Taylor series, for $2<|z|<3$ into one Laurent series, and for $|z|>3$ – into another Laurent series.

Accuracy of the series approximations

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{N} a_n x^n + \underbrace{R_N(x)}_{\text{Remainder}}$$

For calculations it is important to estimate $R_N(x)$, or to decide how many terms N one needs to keep to make sure that $|R_N(x)| < \varepsilon$

Quick reminder: mean value theorem $\quad \int_a^b f(x)\,dx = f(c)(b-a)$

By construction $\quad f^{(N+1)}(x) = R_N^{(N+1)}(x) \Rightarrow$

$$R_N = \underbrace{\int_0^x \int_0^x \cdots \int_0^x}_{N+1 \text{ integrals}} f^{(N+1)}(x)\, d^{N+1}x = f^{(N)}(c) \underbrace{\int_1^x \cdots \int_0^x}_{N \text{ integrals}} x\, d^N x =$$

$$= f^{(N)}(c) \frac{x^{N+1}}{(N+1)!} \qquad 0 < c < x$$

For a Taylor series around x_0 $\quad f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$R_N(x) = f^{(N)}(c) \frac{(x-x_0)^{N+1}}{(N+1)!} \qquad x_0 < c < x$$

To estimate the remainder : $\boxed{|R_N(x)| \leq \max \left[f^{(N+1)}(x) \right]_0^x \frac{x^{N+1}}{(N+1)!}}$

This is the general expression, that always works!

For example $f(x) = \sin x$ (or $\cos x$) $\max |f^{(N+1)}(x)| \leq 1$

so $\underline{|R_N(x)| \leq \frac{x^{N+1}}{(N+1)!}}$

There are a few special cases:

a) Alternating series $S = \sum_{n=0}^{\infty} a_n$. To converge we need $|a_{n+1}| < |a_n|$ and $\lim_{n\to\infty} |a_n| \to 0$

For such a series $R_N = a_{n+1} + a_{n+2} + a_{n+3} + \ldots = \{\text{suppose } a_{n+1} > 0\}$

$= |a_{n+1}| - (|a_{n+2}| - |a_{n+3}|) - (|a_{n+4}| - |a_{n+5}|) - \ldots < |a_{n+1}|$

Thus $|R_N| < |a_{N+1}|$

b) Converging power series for $|x| < 1$

$\sum_{n=0}^{\infty} a_n x^n$, such that $|a_{n+1}| < |a_n|$ (coefficients decrease monothonically)

$$|R_N(x)| = \left| \sum_{n=N+1}^{\infty} a_n x^n \right| \leq \sum_{n=N+1}^{\infty} |a_n| |x|^n < |a_{N+1}| \underbrace{\sum_{n=N+1}^{\infty} |x|^n}_{\text{geometrical progression}}$$

$$|R_N(x)| < \frac{|a_{N+1}| \, |x|^{N+1}}{1-|x|}$$