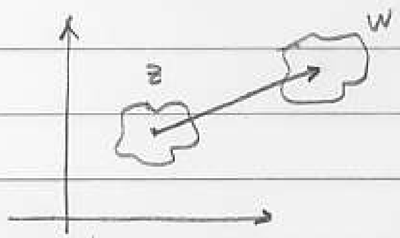
Mapping

As we discussed previously, a complex function may be use to "transform" one region of complex plane into another

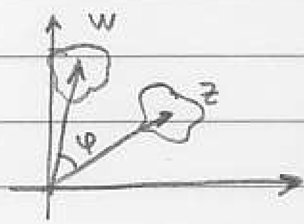
A few simple examples:

1. Translation



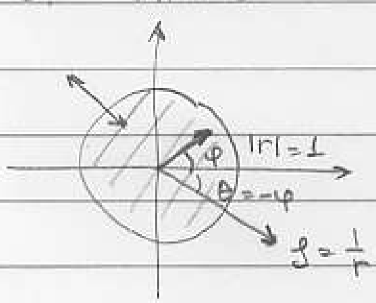
$$W = z + z_0$$
$$(u = x + x_0, \quad v = y + y_0)$$

2. Rotation



$$w = z e^{i\varphi} = (x+iy)(\cos\varphi + i\sin\varphi)$$
$$\begin{cases} u = x\cos\varphi - y\sin\varphi \\ v = x\sin\varphi + y\cos\varphi \end{cases}$$

3. Inversion



$$w = \frac{1}{z}$$
$$\rho e^{i\theta} = \frac{1}{r} e^{-i\varphi}$$

Inside of a unit circle is mapped to its outside.

Or: a circle of radius R mapped to a circle of radius $1/R$

Straight line: $x = x_0$, $y = y_0$

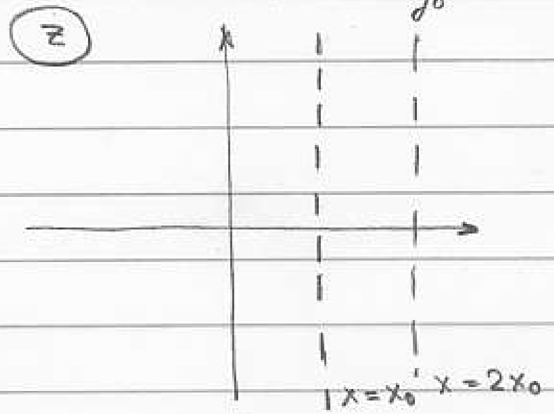
$$w = u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \quad \text{or} \quad x+iy = \frac{u-iv}{u^2+v^2}$$

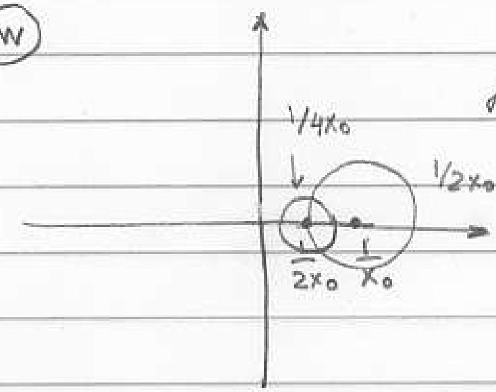
$$x = x_0 = \frac{u}{u^2+v^2} \qquad u^2+v^2 = \frac{u}{x_0} \quad \Rightarrow \quad u^2 - \frac{u}{x_0} + \frac{1}{4x_0^2} + v^2 = \frac{1}{4x_0^2}$$

Or we can write $\left(u - \frac{1}{x_0}\right)^2 + v^2 = \frac{1}{4x_0^2}$

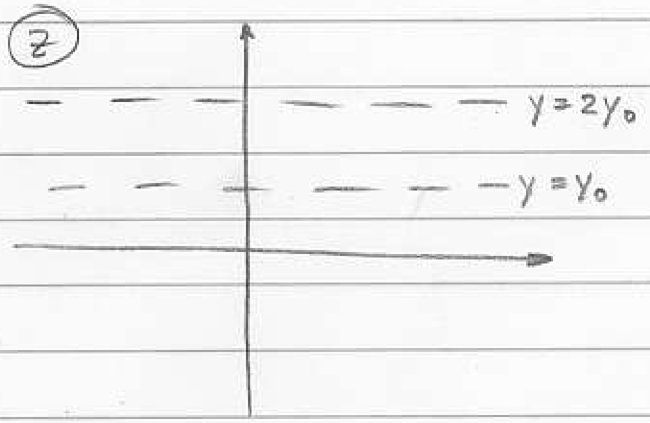
similarly if we repeat same calculations for a horizontal line $y = y_0 = -\frac{v}{u^2+v^2} \Rightarrow$

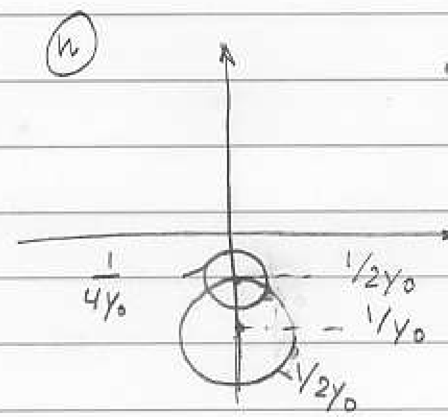
$$u^2 + \left(v + \frac{1}{y_0}\right)^2 = \frac{1}{4y_0^2}$$





circles with the center $1/x_0$ and radius $1/2x_0$

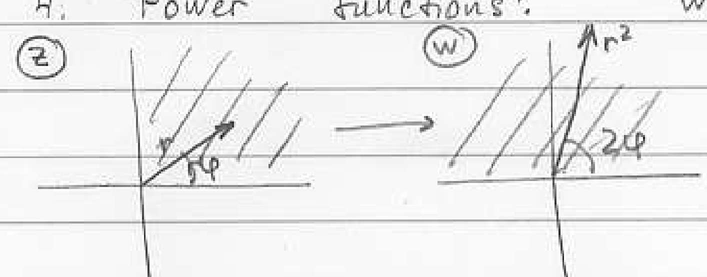




circles with the center $-i\,1/y_0$, and with radius $1/2y_0$

These transformations are 1-to-1 correspondence of complex plane. Other transformations may be more complex

4. Power functions: $w = z^2$ or $w = \sqrt{z}$

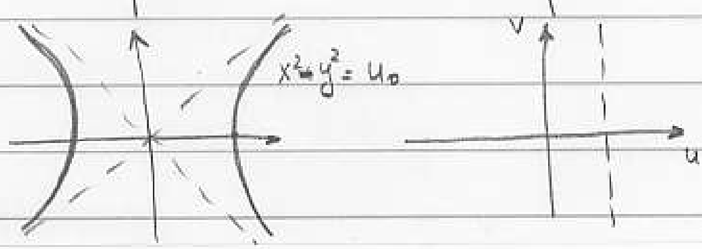


$z = re^{i\varphi}$

$w = r^2 e^{i2\varphi}$

$z = x + iy$

$w = u + iv = x^2 - y^2 + 2ixy$



$u = u_0 = x^2 - y^2 \Rightarrow$ same u_0 corresponds to points (x, y) and $(-x, -y)$

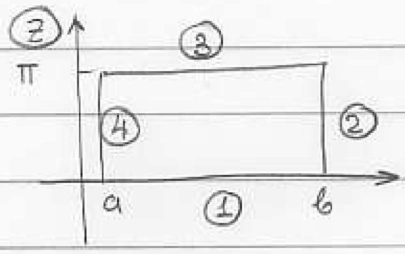
Each quadrant of z-space is mapped into a semi-plane in w-space. Thus, the complex plane in z-space is mapped into two folds of w-space (separated by 2π phase). Such multi-fold surface is called Reimann surface, and each "normal" plane is called a "sheet". Under $w = z^2$ transformation there are two distinguishable sheets of w-space : $[0, 2\pi]$ and $[2\pi, 4\pi]$. However, some function may have infinite amount of sheets. $w = \ln z = \ln r + i\varphi + \underline{i 2\pi n}$ we need infinite # of Reiman sheets of z plane to map complete w-plane.

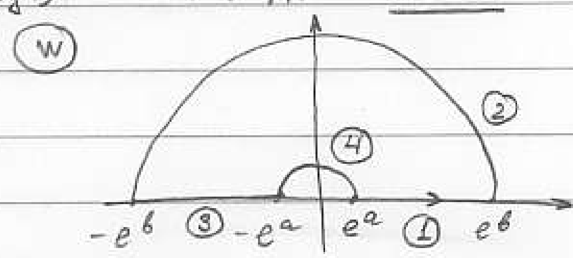
5. Exponent / Logarithm function

$$w = e^z = e^{x+iy} = e^x e^{iy}$$

invert transformation

$$w = \ln z = \frac{1}{2}\ln(x^2+y^2) + i\,\mathrm{atan}\, y/x + \underline{i 2\pi n}$$





$w = e^x e^{iy}$

Exponential function maps a strip in z-plane into a semi-donut-shape in w-plane. Thus, the inverse transformation $w = \ln z$ maps a semi-circles / donuts into strips.

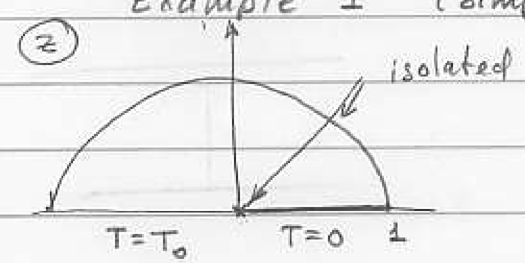
A particular type of mapping - conformal mapping - can be used to solve Laplace equation.
Conformal mapping is the one that conserve the angle b/w lines under transformation. Thus, two orthogonal lines at z-plane will be also orthogonal at w-plane (even though they don't have to be straight)
If $w = f(z)$ is analytic, its mapping is conformal.

It is possible to show that if $g(u,v)$ is a solution of Laplace equation
$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0$, then $h(x,y) = g(u(x,y), v(x,y))$ is a solution of $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$ in x,y coordinates

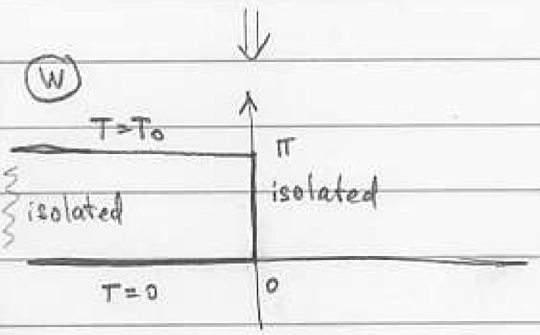
if $w = u + iv = f(z)$

So we can solve a 2D Laplace equation by transforming a "complicated" boundary conditions into "easy" boundary conditions.

Example 1 (simple)



To find the temperature distribution insind such a semicircle.
Transform into a strip using $w = \ln z$ mapping
$x = [0,1] \Rightarrow w = u = [-\infty, 0]$



$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0$$

Boundary conditions $\left.\frac{\partial g}{\partial u}\right|_{u=0} = 0$, $\left.\frac{\partial g}{\partial u}\right|_{u=\infty} = 0$

$g = a + bv$ is a solution

$g = \frac{T_0}{\pi} v$ satisfies all BC

Returning into the z-plane

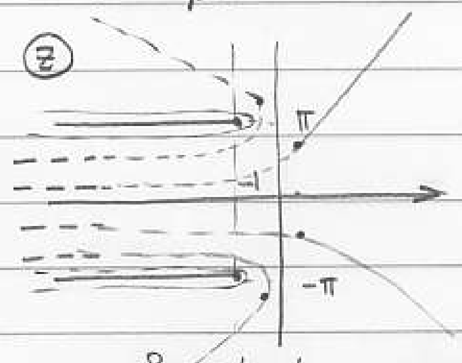
$$w = \ln z = \ln\left(\sqrt{x^2+y^2}\right) + i \tan^{-1}\frac{y}{x} \quad \text{[no need for extra branches]}$$

$$v = \tan^{-1} y/x$$

Thus the solution of the Laplace eqn with BC

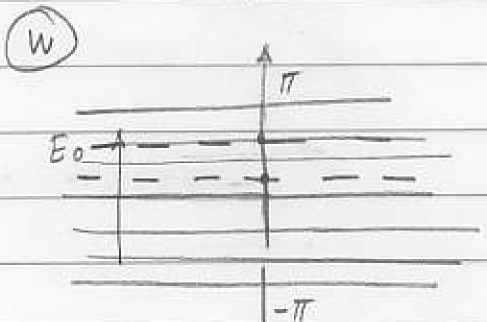
$$h(x,y) = \frac{T_0}{\pi} \tan^{-1} y/x$$

Example 2 (interesting)



Equipotential lines of the edge of the capacitor (of water flow out of the straight tube)

Simplest case — infinite capacitor (or infinite tube)



If the field inside the capacitor is constant, then $\varphi = E_0 \cdot v$

$v = \text{const}$ — equipotential lines

How we can transform "complicated" boundary in z)
into the "easy" boundary in w?

The mapping function we shoul use is

$$z = w + e^{w} \quad \Rightarrow \quad x + iy = u + iv + e^{u}(\cos v + i \sin v)$$

$$x = u + e^{u} \cos v$$

$$y = v + e^{u} \sin v$$

Let's check that this is the right transformation

$w = u \pm i\pi \qquad u = [-\infty, +\infty]$

$$\begin{cases} x = u - e^{u} \\ y = \pm i\pi \end{cases} \qquad u = -\infty \quad x = -\infty, \quad u = 0 \quad x = -1, \quad u = \infty \quad x = -\infty$$

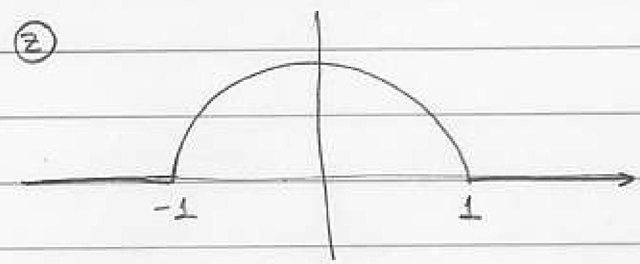
$w = u + iv \qquad |v| < \pi$

$x = u + \cos v \, e^{u} \qquad u = -\infty \quad \begin{matrix} x = -\infty \\ y = v \end{matrix} \; ; \; u = 0 \quad \begin{matrix} x = \cos v \\ y = v + \sin v \end{matrix}$

$y = v + \sin v \, e^{u}$

$u = +\infty \quad \begin{matrix} x = +\infty \; (|v| < \pi/2) \\ x = -\infty \; (|v| > \pi/2) \end{matrix} \quad \begin{matrix} y = +\infty \; v > 0 \\ y = -\infty \; v < 0 \end{matrix}$

Another interesting example



transforms into a flat under $w = z + 1/z$