Mapping

As we discussed previously, a complex function may be used to "transform" one region of complex plane into another.

A few simple examples:

1. Translation

\[ W = z + z_0. \]
\[ (u = x + x_0, \ v = y + y_0) \]

2. Rotation

\[ W = ze^{i\phi} = (x + iy)(\cos \phi + i \sin \phi) \]
\[ \begin{cases} u = x \cos \phi - y \sin \phi \\ v = x \sin \phi + y \cos \phi \end{cases} \]

3. Inversion

\[ W = \frac{1}{z} = \frac{1}{r}e^{-i\theta} \]

Inside of a unit circle is mapped to its outside.

Or: a circle of radius \( R \) mapped to a circle of radius \( 1/R \)

Straight line:
\[ x = x_0, \quad y = y_0 \]
\[ W = u + iv = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} \quad \text{or} \quad \frac{u + iv}{u^2 + v^2} \]
\[ x = x_0 = \frac{u}{u^2 + v^2}, \quad u^2 + v^2 = \frac{u}{x_0} \Rightarrow u^2 - \frac{u}{x_0} + \frac{1}{4x_0^2} + v^2 = \frac{1}{4x_0^2} \]
Dr. we can write 
\[(u - \frac{1}{x_0})^2 + v^2 = \frac{1}{4x_0^2}\]
similarly if we repeat same calculations for a horizontal line 
\[y = y_0 = -\frac{v}{u^2 + v^2} =>\]
\[u^2 + (\nu + \frac{1}{y_0})^2 = \frac{1}{4y_0^2}\]  
circles with 
the center \(1/x_0\) 
and radius 
\(1/2x_0\)

\[x = x_0, x = 2x_0\]

\[y = 2y_0\]
\[-y = y_0, \]  
circles with 
the center 
\(-i/\sqrt{y_0}\), and 
with radius 
\(1/2y_0\)

These transformations are 1-to-1 correspondence of complex plane. Other transformations may be more complex.

H. Power functions: 
\[w = z^2 \text{ or } w = \sqrt{z}\]

\[z = re^{i\theta}\]
\[w = r^2e^{i2\theta}\]
\[z = x + iy\]
\[w = u + iv = x^2 - y^2 + 2ixy\]
\[u = u_0 = x^2 - y^2 => \text{same u_0 corresponds to points (x,y) and (-x,-y)}\]
Each quadrant of \( z \)-space is mapped into a semi-plane in \( w \)-space. Thus, the complex plane in \( z \)-space is mapped into two folds of \( w \)-space (separated by \( 2\pi \) plane).

Such multi-fold surface is called Reimann surface, and each "normal" plane is called a "sheet".

Under \( w = z^2 \) transformation there are two distinguishable sheets of \( w \)-space: \([0, 2\pi]\) and \([2\pi, 4\pi]\).

However, some function may have infinite amount of sheets. \( w = \ln z = \ln r + i\varphi + i2\pi n \)

we need infinite # of Reimann sheets of \( z \) plane to map complete \( w \)-plane.

5. Exponent / Logarithm function

\[
w = e^z = e^{x + iy} = e^x e^{iy}
\]

invert transformation

\[
w = \ln z = \frac{1}{2} \ln \left( x^2 + y^2 \right) + i \tan^{-1} \frac{y}{x} + i2\pi n
\]

Exponential function maps a strip in \( z \)-plane into a semi-donut-shape in \( w \)-plane.

Thus, the inverse transformation \( w = \ln z \) maps a semi-circles / donuts into strips.
A particular type of mapping—conformal mapping—can be used to solve Laplace equation. Conformal mapping is the one that conserve the angle b/w lines under transformation. Thus, two orthogonal lines at \( z \)-plane will be also orthogonal at \( w \)-plane (even though they don't have to be straight).

If \( w = f(z) \) is analytic, its mapping is conformal.

It is possible to show that if \( g(u,v) \) is a solution of Laplace equation
\[
\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0,
\]
den \( h(x,y) = g(u(x,y), v(x,y)) \) is a solution of
\[
\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \text{ in } x,y \text{ coordinates}
\]
if \( w = u + iv = f(z) \).

So we can solve a 2D Laplace equation by transforming a "complicated" boundary conditions into "easy" boundary conditions.

**Example 1 (simple)**

\[ z = 0 \]

To find the temperature distribution inside such a semicircle, transform into a strip using

\[ w = \ln z \text{ mapping} \]

\[ x = [0,1] \Rightarrow w = u = [-\infty,0] \]

\[ z = \pi \]

Boundary Conditions \( \frac{\partial g}{\partial u} |_{u=0} = 0, \frac{\partial g}{\partial u} |_{u=-\infty} = 0 \)

\( g = a + bv \) is a solution

\( g = \frac{T_0}{\pi} \nu \) satisfies all BC
Returning into the $z$-plane

$w = \ln z = \ln (x^2 + y^2) + i \tan^{-1} \frac{y}{x}$ [no need for extra branches]

$\nu = \tan^{-1} \frac{y}{x}$

Thus the solution of the Laplace eqn with BC

$h(x, y) = \frac{T_0}{\pi} \tan^{-1} \frac{y}{x}$

Example 2 (interesting)

Equi-potential lines of the edge of the capacitor (or water flow out of the straight tube)

Simplest case - infinite capacitor (or infinite tube)

If the field inside the capacitor is constant, then $\nu = E_0 \cdot \nu$

If the water flows into speed $v$  

How we can transform "complicated" boundary in $z \rightarrow$ into the "easy" boundary in $w$

The mapping function we shall use is

$z = w + e^w \Rightarrow x + iy = u + iv + e^u (\cos v + i \sin v)$

$x = u + e^u \cos v$

$y = v + e^u \sin v$

Let's check that this is the right transformation
\[ w = u \pm i\pi \quad u = \left[ -\infty, +\infty \right] \]

\[
\begin{align*}
x &= u = e^u \quad (u < 0) \quad u = -\infty \quad x = -\infty \quad u = 0 \quad x = -1 \quad u = \infty \quad x = -\infty \\
y &= \pm i\pi
\end{align*}
\]

\[ w = u + iv \quad |v| < \pi \]

\[
\begin{align*}
x &= u + \cos v e^u \quad u = -\infty \quad x = -\infty \quad u = 0 \quad x = \cos v \\
y &= v + \sin v e^u
\end{align*}
\]

\[
\begin{align*}
u &= +\infty \quad x = +\infty \quad (|v| < \frac{\pi}{2}) \\
x &= -\infty \quad (|v| > \frac{\pi}{2}) \\
y &= -\infty \quad v < 0
\end{align*}
\]

Another interesting example

Transforms into a flat under \( w = z + \frac{1}{2} \)