Lecture #18
2009

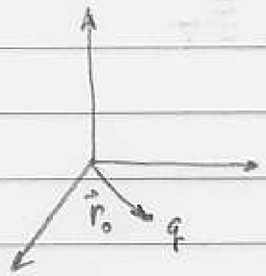
Nonlinear PDE

Poisson equation $\nabla^2 u = f(\vec{r})$

The equation is nonlinear, since the sum of two solutions is <u>not</u> a solution.

Poisson equation describes the electrostatic potential $\varphi(\vec{r})$ of the charge density $f(\vec{r}) = -\rho(\vec{r})/\varepsilon_0$.

And we know how to solve this equation!



Point charge at $\vec{r} = \vec{r}_0$

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{|\vec{r}-\vec{r}_0|}$$

Several charges

$$\varphi(\vec{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0} \frac{1}{|\vec{r}-\vec{r}_i|}$$

Continuous charge distribution

$$q_i \to \rho(\vec{r}\,')\, d^3\vec{r}\,' \qquad \sum_{i=1}^{N} \to \int_V$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\vec{r}\,')\, d^3\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} = \int_V \rho(\vec{r}\,')\, \varphi_0(\vec{r},\vec{r}\,')\, d^3\vec{r}\,'$$

$\varphi(\vec{r})$ is a solution of $\nabla^2\varphi = -\rho(\vec{r})/\varepsilon_0$, and

$\varphi_0(\vec{r},\vec{r}\,')$ is a solution for a unit point charge:

$$\rho_0(\vec{r}) = q\,\delta(\vec{r}-\vec{r}\,') \Rightarrow \varphi_0(\vec{r},\vec{r}\,') = \frac{q}{4\pi\varepsilon_0} \frac{1}{|\vec{r}-\vec{r}\,'|}$$

that is the solution of

$$\nabla_r^2 \varphi_0(\vec{r},\vec{r}\,') = -q/\varepsilon_0\, \delta(\vec{r}-\vec{r}\,')$$

Thus, we can generalize that a solution of $\nabla^2 u = f(\vec{r})$ can be found as

$$u(\vec{r}) = \int f(\vec{r}\,') \, G(\vec{r},\vec{r}\,') \, d^3\vec{r}\,'$$

where $G(\vec{r},\vec{r}\,')$ is the solution of equation

$$\nabla_{\vec{r}}^2 \, G(\vec{r},\vec{r}\,') = - \delta(\vec{r}-\vec{r}\,')$$

$G(\vec{r},\vec{r}\,')$ is called Green's function. Once the Green's function is found for particular equation $\hat{\mathcal{L}} u = f(\vec{r})$, such that $\hat{\mathcal{L}}\, G(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,')$, then the solution for <u>any</u> function $f(\vec{r})$ is just $u(\vec{r}) = \int f(\vec{r}\,') \, G(\vec{r},\vec{r}\,') d^3\vec{r}\,'$

We identified the Green's function for Poisson equation: $\nabla^2 u = f(\vec{r})$ $\quad G(\vec{r},\vec{r}\,') = \frac{1}{4\pi} \frac{1}{|\vec{r}-\vec{r}\,'|}$

Nonlinear Helmholtz equation

$$(\nabla^2 + \vec{k}_0)\, u(\vec{r}) = f(\vec{r})$$

The equation for the Green's function $(\nabla^2 + k^2)\, G(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,')$

$$G(\vec{r},\vec{r}\,') = \frac{1}{(2\pi)^3} \int \frac{e^{i\vec{k}(\vec{r}-\vec{r}\,')}}{k^2 - k_0^2} \, d^3\vec{k} = \frac{1}{4\pi} \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|r-r'|}$$

The next question is how to deal with the boundary conditions:

a) We can use the "usual" approach: if $u(\vec{r})$ is the solution of nonlinear eqn $\mathcal{L} u = f(\vec{r})$, and $v(\vec{r})$ is a solution of corresponding linear eqn $\mathcal{L} v = 0$, then $u+v$ is the solution of the nonlinear eqn. Thus we can add a necessary combination of the solutions of the linear eqn to satisfy the boundary conditions

However, if we need to solve the same nonlinear eqn with <u>same</u> boundary conditions for <u>different</u> nonlinear parts $f(\vec{r})$, we may modify the Green's function instead to take into account the boundary conditions.

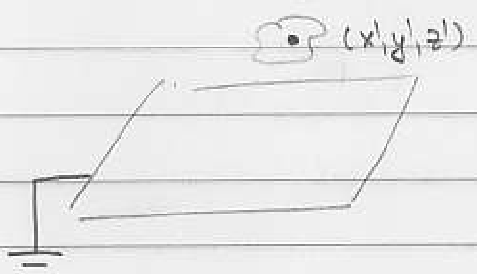
Indeed, if $\mathcal{L}G(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,')$, and $v(\vec{r})$ is the solution of $\mathcal{L}v(\vec{r}) = 0$, then

$G'(\vec{r},\vec{r}\,') = G(\vec{r},\vec{r}\,') + v(\vec{r})$ is also the Green's function. Thus, by combining the known Greens function with appropriate combination of $\{v(\vec{r})\}$ we can satisfy the boundary condition for any nonlinear function $f(\vec{r})$

For example: if $G'(\vec{r},\vec{r}\,')\Big|_{\vec{r},\,\text{surface}} = 0$

$$u(\vec{r})\Big|_{\text{surface}} = \int f(\vec{r}\,')\, G'(\vec{r},\vec{r}\,')\Big|_{\text{surface}} d r' = 0 \quad \text{for any } f(\vec{r}\,')$$

More examples from electrostatics:



a) Charge distribution on top of grounded conducting plane

$$\nabla^2 u = -\frac{\rho(\vec{r})}{\varepsilon_0}$$

boundary condition $u(x,y,z=0)=0$.

We know the Green's function for the Poissone equation

$$G(\vec{r},\vec{r}\,') = \frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{4\pi}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

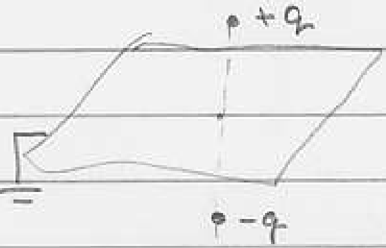
To satisfy the boundary conditions we want $G'(\vec{r},\vec{r}\,') = 0$ for $z=0$

Notice, that our Greens function corresponds to the charge in the above the plane, and the corresponding solution for any charge below the plane will satisfy Laplace eqn in the upper space

Thus $V(\vec{r}) = \frac{1}{4\pi} \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$ satisfy $\nabla^2 V = 0$ at $z \geq 0$.

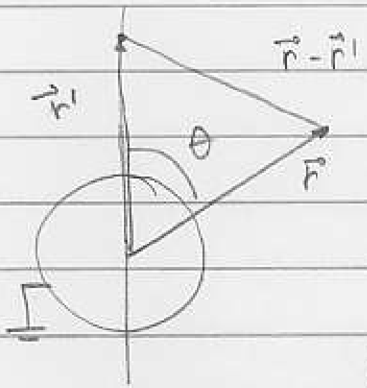
$$G'(\vec{r},\vec{r}') = \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right)$$

That corresponds to the physics solution we all know:



a point charge above the grounded plane is equivalent to the image charge of opposite side below the plane.

Another example: charge distribution next to grounded sphere



$$\nabla^2 \varphi = -\frac{1}{\varepsilon_0}\rho(\vec{r})$$

$$\varphi|_{r=a} = 0$$

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi}\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}}$$

We know the solutions of Laplace eqn in spherical geometry $V(\vec{r}) = \sum_{\ell=0}^{\infty}\frac{A_\ell}{r^{\ell+1}}P_\ell(\cos\theta)$

(no φ-dependence $\rightarrow m=0$)

To satisfy the boundary conditions

$$G'(\vec{r},\vec{r}') = \frac{1}{4\pi}\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} + \sum_{\ell=0}^{\infty}\frac{A_\ell}{r^{\ell+1}}P_\ell(\cos\theta) \Big|_{r=a} = 0$$

$$G'(a,\vec{r}') = \frac{1}{4\pi}\frac{1}{\sqrt{a^2+r'^2-2ar'\cos\theta}} + \sum_{\ell=0}^{\infty}\frac{A_\ell}{a^{\ell+1}}P_\ell(\cos\theta) = 0$$

$$\frac{1}{\sqrt{a^2+r'^2+2ar'\cos\theta}} = \frac{1}{r'}\frac{1}{\sqrt{1+(a/r')^2-2(a/r')\cos\theta}} = \frac{1}{r'}\sum_{\ell=0}^{\infty}\left(\frac{a}{r'}\right)^{\ell} P_\ell(\cos\theta)$$

$$G'(a,r') = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}\frac{a^\ell}{r'^{\ell+1}}P_\ell(\cos\theta) + \sum_{\ell=0}^{\infty}\frac{A_\ell}{a^{\ell+1}}P_\ell(\cos\theta) = 0$$

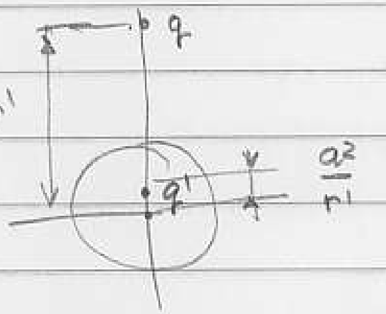
$$A_\ell = -\frac{1}{4\pi}\frac{a^{2\ell+1}}{r'^{\ell+1}}$$

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi}\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} - \frac{1}{4\pi}\frac{a}{rr'}\sum_{\ell=0}^{\infty}\left(\frac{a^2}{rr'}\right)^{\ell}P_\ell(\cos\theta) =$$

$$= \frac{1}{4\pi}\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} - \frac{1}{4\pi}\frac{a}{rr'}\frac{1}{\sqrt{1+\frac{a^4}{(rr')^2}-2\frac{a^2}{rr'}\cos\theta}} =$$

$$= \frac{1}{4\pi}\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} - \frac{1}{4\pi}\frac{a}{r'}\frac{1}{\sqrt{r^2+\left(\frac{a^2}{r'}\right)^2-2r\left(\frac{a^2}{r'}\right)\cos\theta}} =$$

$$= \frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}'|} - \frac{1}{4\pi}\frac{a/r'}{|\vec{r}-\vec{R}'|} \qquad \vec{R}' = \frac{a^2}{r'^2}\vec{r}'$$



$q' = -\frac{a}{r'}q$, and the image charge is positioned inside the sphere at $R' = \frac{a^2}{r'}$ from the center.