Lecture # 16
2009

Time - dependent PDEs (partial differential eqns)

Diffusion equation

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$ [often written as $D\nabla^2 u = \frac{\partial u}{\partial t}$; where D is diffusion constant]

We will use this equation to describe the temperature evolution in various geometries.

From the mathematical point of view notice that if $u(x,t)$ is the solution of diffusion eqn, we can add any solution $v(x)$ of Laplas eqn ($\nabla^2 v = 0$) to it! $\left(\frac{\partial^2(u+v)}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}\right)$

We will consider the following sequence of events:

1. System is in steady state (Laplas equation) → initial condition → 2. Something changes (Diffusion equation) →
3. System evolves to new equilibrium (Laplas equation) → boundary condition.

Now in addition to the boundary conditions we have to add initial condition — describing the state of the system before the pertrubation.

Since we expect that the system will evolve to new equilibrium, we can expect that our solution will consist of two parts: time-independent part $u_{fin}(x)$ that correspons to the solution of the Laplas eqn for the <u>final</u> boundary conditions, and the time-dependent (decaying) part $u_{diff}(x,t)$ that describes the evolution of the system.

Final solution $u(x,t) = u_{fin}(x) + u_{diff}(x,t)$

Note that $u_{fin}(x)$ takes care of the Boundary conditions for $u(x,t)$

Thus $u_{diff}(x,t)$ should have <u>zero</u> boundary conditions.

Then the initial state of the system $u_{ini}(x)$ must be a solution of Laplace equation with the initial boundary conditions, and

$$u_{ini}(x) = u_{fin}(x) + u_{diff}(x, t=0)$$

1D case: $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial u}{\partial t}$

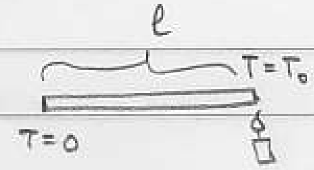
Separating the variables: $u(x,t) = X(x)\,T(t)$

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{\dot{T}}{T} = -k^2$$

The separation constant must be negative to ensure that the time-dependent part of the solution decays in time

$$\dot{T} = -k^2\alpha^2 T \qquad T = e^{-k^2\alpha^2 t}$$

$$u_{dif}(x,t) = \sum_k A_k X_k(x)\, e^{-k^2\alpha^2 t}$$

Example 1: Initial state:  $t<0$

Final state:  $t>0$

Initial temperature distribution: $X''_{ini} = 0 \qquad X_{ini} = a + bx$

$$X_{ini} = T_0\, x/\ell$$

Final temperature distribution: $X_{fin} = 0$

$$X'' = -k^2 X \qquad X = A\cos kx + B\sin kx$$

Since $X(0) = X(\ell) = 0 \qquad X = B_n \sin\frac{\pi n x}{\ell} \qquad n = 0, 1, 2..$

Thus $u(x,t) = \displaystyle\sum_{n=1}^{\infty} B_n \sin\frac{\pi n x}{\ell}\, e^{-\frac{\pi^2 n^2 \alpha^2}{\ell^2}t}$

and

$$u(0,t) = T_0\frac{x}{\ell} = \underline{\frac{2T_0}{\pi}}\sum_{n=1}^{\infty}\underline{\frac{(-1)^{n-1}}{n}}\sin\frac{\pi n x}{\ell} = \sum_{n=1}^{\infty}\underline{B_n}\sin\frac{\pi n x}{\ell}$$

Final solution: $u(x,t) = \dfrac{2T_0}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{\pi n x}{\ell}\, e^{-\frac{\pi^2 n^2 \alpha^2}{\ell^2}t}$

Example 2: Reverse the situation

Initial: 

Final: 

$u_{ini}(x) = 0 \qquad\qquad u_{fin}(x) = T_0\, x/\ell$

$$u(x,t) = u_{fin} + u_{dift} = T_0\, x/\ell + \sum_k A_k X_k(x)\, e^{-k^2\alpha^2 t}$$

It is clear that to obey final boundary conditions

$$X_k(0) = X_k(\ell) = 0 \quad \Rightarrow \quad X_k(x) = \sin\frac{\pi n x}{\ell} \qquad n = 0, 1, 2, \ldots$$

$$u(x,t) = T_0\, x/\ell + \sum_{n=1}^{\infty} A_n \sin\frac{\pi n x}{\ell}\, e^{-\frac{\pi^2 n^2 \alpha^2}{\ell^2} t}$$

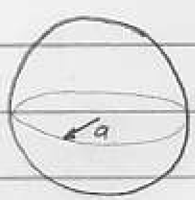
and

$$T_0\, x/\ell + \sum_{n=1}^{\infty} A_n \sin\frac{\pi n x}{\ell} = 0 \qquad \text{(initial conditions)}$$

$$\frac{T_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{\pi n x}{\ell} + \sum_{n=1}^{\infty} A_n \sin\frac{\pi n x}{\ell} = 0 \Rightarrow A_n = -\frac{T_0}{\pi}\,\frac{(-1)^{n-1}}{n}$$

Final distribution: $$u(x,t) = T_0\, x/\ell - \sum_{n=1}^{\infty}\frac{T_0}{\pi}\,\frac{(-1)^{n-1}}{n}\sin\frac{\pi n x}{\ell}\, e^{-\frac{\pi^2 n^2 x^2}{\ell^2}}$$

3D - case



Sphere

$$T_{ini} = T_0 \sin^2\theta \cos\theta \cos 2\varphi$$

$$T_{fin} = 0$$

From the last lecture: a general form of the solution of the Laplace eqn inside the sphere is

$$u(r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (r/a)^{\ell} P_\ell^m(\cos\theta)\,(A_{\ell m}\cos m\varphi + B_{\ell m}\sin m\varphi)$$

$$T_{ini} = T_0 \overbrace{\sin^2\theta\cos\theta}\ \underset{\substack{\uparrow \\ m=2}}{\cos 2\varphi}$$

→ must present $\sin^2\theta\cos\theta$ as a sum of all $P_\ell^2(\cos\theta)$

luckily $P_3^2(\cos\theta) = 15 \sin^2\theta \cos\theta$

Thus $T_{ini} = \frac{1}{15} T_0\, P_3^2(\cos\theta) \cos 2\varphi \Rightarrow$ and

$$u_{ini}(r,\theta,\varphi) = \frac{1}{15} T_0\, (r/a)^3\, P_3^2(\cos\theta)\cos 2\varphi, \quad u_{fin}(r,\theta,\varphi) = 0$$

Diffusion eqn in 3D (spherical geometry)

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} = -k^2 \quad \text{for the same reason as before}$$

$$u(\vec{r},t) = \sum_k A_k F_k(\vec{r})\, e^{-\alpha^2 k^2 t}$$

$$\Downarrow$$

$$\nabla^2 F = -k^2 F \qquad \text{Helmholtz equation for spatial coordinates}$$

We should solve this equation following the same steps as for Laplas equation before:

Separating the variables: $F(r,\theta,\varphi) = R(r)\,P(\theta)\cdot\Phi(\varphi)$

$$\frac{1}{R}\frac{1}{r^2}\frac{d^2}{dr^2}(r^2 R') + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dP}{d\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} = \underline{-k^2}$$

Angular part of the solution is the same as for Laplace eqn

$$\rightarrow P_\ell^m(\cos\theta)\,(A_{\ell m}\cos m\varphi + B_{\ell m}\sin m\varphi)$$

Radial part is different:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \ell(\ell+1)R + k^2 r^2 R = 0$$

Solutions are the <u>spherical Bessel functions</u> $j_\ell(kr)$

In order to obey the zero boundary condition

$$j_\ell(ka) = 0 \implies j_\ell\left(k_i^{(\ell)}\frac{r}{a}\right) \qquad k_i = k_i^{(\ell)}/a$$

Initial conditions

$$u_{diff}(r,\theta,\varphi,t=0) = \sum_{i,\ell,m} j_\ell\left(k_i^{(\ell)}\frac{r}{a}\right) P_\ell^m(\cos\theta)\,(A_{\ell m i}\cos m\varphi + B_{\ell m i}\sin m\varphi)$$

$$= \frac{1}{15}T_0\,(r/a)^3\,P_3^2(\cos\theta)\cos 2\varphi \implies \ell=3,\ m=2$$

$$\sum_i A_i\, j_3\left(k_i^{(3)}\frac{r}{a}\right) = \frac{1}{15}T_0\,(r/a)^3$$

can use orthogonality to find the coefficients A_i and the general solution is

$$u(r,\theta,\varphi,t) = \sum_{i=1}^{\infty} A_i\, j_3\left(k_i^{(3)}\frac{r}{a}\right) P_3^2(\cos\theta)\cos 2\varphi\; e^{-\alpha^2 (k_i^{(3)})^2/a^2\, t}$$

Orthogonality for spherical Bessel functions

$$\int_0^1 j_\ell(k_i^{(\ell)}x)\, j_\ell(k_j^{(\ell)}x)\, x^2 dx = \frac{\delta_{ij}}{2}\, j_{\ell+1}^2 (k_i^{(\ell)})$$

or $$\int_0^a j_\ell (k_i^{(\ell)} \tfrac{r}{a})\, j_\ell(k_j^{(\ell)} \tfrac{r}{a})\, r^2 dr = \delta_{ij} \frac{a^3}{2} j_{\ell+1}^2 (k_i^{(\ell)})$$

$$\sum_{i=1}^{\infty} A_i \int_0^a j_3(k_i \tfrac{r}{a})\, j_3(k_j \tfrac{r}{a})\, r^2 dr = \frac{1}{15} T_0 \int_0^a (r/a)^3 j_3(k_j \tfrac{r}{a})\, r^2 dr$$

$$A_j \frac{a^3}{2} j_4^2(k_j) = \frac{T_0}{15} a^3 \frac{1}{k_j^6} \int_0^{k_j} x^5 j_3(x)\, dx$$

Recurrence relations for spherical Bessel functions

$$\frac{d}{dx}[x^n j_n(x)] = x^{n+1} j_{n-1}(x)$$

thus

$$x^5 j_3(x) = \frac{d}{dx}[x^4 j_4(x)]$$

$$A_j \frac{\cancel{a^3}}{2} j_4^{\cancel{2}}(k_j) = \frac{1}{15} T_0 \cancel{a^3} \frac{1}{k_j^6} \cdot k_j^4 \cdot \cancel{j_4(k_j)}$$

$$A_j = \frac{2T_0}{15} \frac{1}{k_j^2 j_4(k_j)}$$

$$u(r,\theta,\varphi,t) = \sum_{i=1}^{\infty} \frac{2T_0}{15} \frac{j_3(k_i r/a)}{k_i^2 j_4(k_i)} \sin^2\theta \cos\theta \cos 2\varphi \, e^{-\frac{d^2 k_i^2}{a^2} t}$$

where k_i – are the zeros of $j_3(x)$