Laplace equation in cylindrical coordinates

\[ \nabla^2 u = 0 \]  (Boas ch 10-9)

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \]

Separate the variables:

\[ u(r, \theta, z) = R(r) \Phi(\theta) Z(z) \]

\[ \frac{1}{r R} \frac{d}{dr} \left( r R' \right) + \frac{1}{r^2} \frac{\Phi''}{\Phi} + \frac{Z''}{Z} = 0 \]

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

\[ J = \nabla \times \mathbf{B} \]

**Infinite cylinder at constant temperature**

\[ \Phi = Z = 0 \]

\[ \frac{d}{dr} (rR') = 0 \quad rR' = \text{constant} = C_1 \quad R = C_2 + C_1 \ln r \]

Inside: solution must be limited: \( C_1 = 0 \)

\[ R(r) = C_2 = T_0 \]

But when calculating the temperature outside we run into a problem, since \( \ln r \) diverges at \( r \to 0 \).

There is a physical reason for that --- we assumed that the cylinder is indefinitely long, and at some distance away this approximation will break. To resolve this mathematically, we will add one more boundary condition:

at some distance \( b \gg a \): \( T(b) = T_b \) (background temperature)

so for the outside region:

\[ \int C_2 + C_1 \ln b \, \text{d}a = T_0 \quad \Rightarrow \quad C_1 = \frac{T_0 - T_b}{\ln b} \]

\[ \int C_2 + C_1 \ln b \, \text{d}a = T_b \quad \Rightarrow \quad C_2 = -C_1 \ln b + T_b \]

Thus \( u(r) = T_0 + C_1 \ln \frac{r}{a} = T_0 + \frac{T_0 - T_b}{\ln b} \ln \frac{r}{a} \)
Cylindrical "capacitor"

Boundary conditions: \( T(r=a) = T_0 \), \( r < a \) \n
There is still no \( z \)-dependence \((z = 0)\) \n
but now there is \( \varphi \)-dependence \( \frac{1}{rK} \frac{d}{dr} (rR') + \frac{1}{r^2} \left( \frac{d}{d\varphi} \frac{d\varphi}{d\varphi} \right) = 0 \)

\( \Phi'' = -m^2 \Phi \Rightarrow \Phi = A_m \cos m\varphi + B_m \sin m\varphi \)

Since physically \( \varphi \) and \( \varphi + 2\pi \) describe the same point in space, we require \( \Phi(p) = \Phi(p + 2\pi) \Rightarrow m = \text{integer} \)

Equation for \( R \): \( \frac{1}{rR} \frac{d}{dr} (rR') - \frac{m^2}{r^2} = 0 \Rightarrow r \frac{d}{dr} (rR') - m^2 R = 0 \)

Solutions \( R = r \pm m \) or \((1, \ln r)\) for \( m = 0 \)

Solution: \( u(r, \varphi) = C_1 + C_2 \ln r + \sum_{m=1}^{\infty} (A_m \cos m\varphi + B_m \sin m\varphi) (C_m r^m + D_m \frac{1}{r^m}) \)

This is the general form, and now we can simplify that using the boundary conditions, and requiring finite solution:

a. Inside: \( C_2 = 0, D_m = 0 \)

b. Outside: \( C_2 = 0, C_m = 0 \)

c. \( \text{BC are anti-symmetric in} \ \varphi \rightarrow A_m = 0 \)

d. at \( r = a \) \( u(r, \varphi) \to T_0 \), and at \( r = 0 \) \( u(0, \varphi) = T_0 \) \( \Rightarrow C_1 = T_0 \)

Thus \( u_{\text{inside}}(r, \varphi) = T_0 + \sum_{m=1}^{\infty} B_m \sin m\varphi \)

outside

To find \( B_m \), we need to decompose the boundary function in the sine Fourier series \( T_0 = T_0 + \sum_{m=1}^{\infty} T_m \sin m\varphi \)

\( T_m = \frac{2}{\pi} \int \limits_{-\pi}^{\pi} \Delta T(\varphi) \sin m\varphi d\varphi = \frac{2\Delta T}{m\pi} (\cos m\varphi) \bigg|_{0}^{\pi} = \frac{4\Delta T}{m\pi} \left( \frac{1}{2} \right) = \frac{2\Delta T}{m\pi} \)

\( u(0, \varphi) = T_0 + \sum_{m=1}^{\infty} B_m \sin m\varphi \Rightarrow B_m = T_m = \frac{4\Delta T}{m\pi} \) \( m = 1, 3, 5, \ldots \)

inside/outside \( (r, \varphi) = T_0 + \sum_{m=1, \text{odd}}^{\infty} \frac{4\Delta T}{m\pi} \sin m\varphi \left( \frac{r}{a} \right) \)
Semi-infinite cylinder

\[ \frac{1}{rR} \frac{d}{dr} (rR') + \frac{1}{r^2} \frac{\Phi''}{\Phi} + \frac{n^2}{k^2} = 0 \]

\[ \Phi(z) = e^{-kz} \text{ exponential} \]

\[ \Phi(z) \text{ is oscillatory} \]

\[ \Phi(R) = A_n \cos n\phi + B_n \sin n\phi \quad n = \text{integer} \]

\[ \frac{1}{rR} \frac{d}{dr} (rR') - \frac{n^2}{k^2} = 0 \]

\[ r^2 R'' + rR' + (k^2 r^2 - n^2) R = 0 \text{ Bessel eqn.} \]

Solutions are:

\[ R(r) = J_n(kr) \quad \text{for convenience } k \to k/a \]

\[ R(r) = J_n(kr/a) \]

**Link** \( (r, \psi, z) = J_n(kr/a) (A_n \cos n\phi + B_n \sin n\phi) e^{-kz/a} \)

Now let's take into account the boundary conditions:

\[ u = 0 \text{ at } r = a \Rightarrow J_n(k) = 0 \Rightarrow k = \lambda^{(n)}(a) \quad \text{zeros of the Bessel function} \]

\[ u(r, \psi, z) = \sum \lambda^{(n)}(a) J_n(d^{(n)} r/a) (A_n \cos n\phi + B_n \sin n\phi) e^{-kz/a} \]

Then \( u(r, \psi, z = 0) = f(r, \psi) = \sum \lambda^{(n)}(a) J_n(d^{(n)} r/a) (A_n \cos n\phi + B_n \sin n\phi) \)

So to find the coefficients \( A_n,B_n \) one has to decompose \( f(r, \psi) \) into the Fourier series in \( \phi \) and Bessel series in \( r \).

**Simple example** \( \Rightarrow \psi(r, \phi) = T_0 \) \( A_n = B_n = 0 \) for \( n > 0 \)

\[ B_0 = 0 \]

\[ \sum A_n J_0(d^{(n)} r/a) = T_0 \]

\[ J_0(d^{(n)} r/a) \text{ and integrate} \]

Using orthogonality condition of the Bessel functions:

\[ A_n \frac{a^2}{2} J_1^2(d^{(n)} r/a) = T_0 \int J_0(d^{(n)} r/a) r dr = T_0 a^2 \frac{1}{d^{(n)} J_1(d^{(n)} r/a)} \]

\[ A_n = \frac{2T_0}{d^{(n)} J_1(d^{(n)} r/a)} \]

\[ u(r, \psi, z) = \sum \frac{2T_0}{d^{(n)} J_1(d^{(n)} r/a)} J_0(d^{(n)} r/a) e^{-kz/a} \]
Laplace equation in spherical coordinates

\[ \nabla^2 u = 0 \]

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0 \]

\[ u(r, \theta, \phi) = R(r) \Phi(\theta) \Psi(\phi) \]

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Phi}{d\theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{d^2 \Psi}{d\phi^2} = 0 \]

as before: \[ \Phi = A_m \cos \theta + B_m \sin \theta \]

\[ m = 0, 1, 2, \ldots \]

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR'}{dr} \right) + \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Phi'}{d\theta} \right) - \frac{m^2}{\sin \theta} \frac{\Phi'}{\Phi} = 0 \]

\[ \ell (\ell + 1) \]

\[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP}{d\theta} \right) - \frac{m^2}{\sin \theta} P + \ell (\ell + 1) P = 0 \]

solutions: associate Legendre functions \( P^m_\ell (\cos \theta) \)

Radial part: \[ \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \ell (\ell + 1) R = 0 \]

\[ R = r^2 \text{ or } r^{\ell - 2} \]

"Inside" and "outside" solutions

\[ u_{\text{in}}(r, \theta, \phi) = \sum_{\ell = 0}^{\infty} \sum_{m = -\ell}^{\ell} r^\ell P^m_\ell (\cos \theta) (A_m \cos \phi + B_m \sin \phi) \]

or \[ r^\ell Y_{\ell} (\theta, \phi) \]

\[ u_{\text{out}}(r, \theta, \phi) = \sum_{\ell = 0}^{\infty} \sum_{m = -\ell}^{\ell} \frac{1}{r^{\ell + 1}} P^m_\ell (\cos \theta) (A_m \cos \phi + B_m \sin \phi) \]

or \[ \frac{1}{r^{\ell + 1}} Y_{\ell} (\theta, \phi) \]

where \[ Y_{\ell} (\theta, \phi) = (-1)^m \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P^m_\ell (\cos \theta) e^{im\phi} \]

spherical harmonics.
(1) Spherical capacitor

Boundary conditions:
In general, the values of the solution
\( u(r=a, \theta, \phi) = f(\theta, \phi) \) is given, and
it has to be decomposed into
the series of spherical harmonics.

In our case there is no \( \phi \)-dependence
on \( \phi \) in our boundary conditions
\((m=0)\). Moreover, \( f(\cos \theta) \) is odd, so
we expect only odd \( \ell \) to appear
in the final summation.

Solution inside:\n\[
\begin{align*}
  u(r, \theta) &= \sum_{\ell=0}^{\infty} A_\ell \left( \frac{r}{a} \right) \ell \ P_\ell (\cos \theta) \\
  &\quad (\text{even})
\end{align*}
\]

outside \( u(r, \theta) = \sum_{\ell=0}^{\infty} A_\ell \left( \frac{a}{r} \right)^{\ell+1} \ P_\ell (\cos \theta) \)

To find the coefficient \( A_\ell \), we have to decompose \( f(\cos \theta) \)
into the series of \( P_\ell (\cos \theta) \) [or \( f(x) \) into \( -P_\ell (x) \)],

\[
  u(a, \theta) = \sum_{\ell=0}^{\infty} A_\ell \ P_\ell (\cos \theta) = f(\cos \theta) \times P_\ell (x)
\]

using orthogonality of \( P_\ell \):
\[
  A_\ell \frac{2}{2\ell+1} = \int_{-1}^{1} f(x) P_\ell (x) \, dx
\]

\[
  = 2 \int_{0}^{V_0} P_\ell (x) \, dx = \frac{x}{2\ell+1} P_\ell = P_\ell' - P_\ell \Rightarrow A_\ell = \frac{2V_0}{2\ell+1} \left( \frac{P_{\ell+1}(0)}{2\ell+1} - \frac{P_{\ell-1}(0)}{2\ell+1} \right)
\]

\[
  = \frac{2V_0}{2\ell+1} \left( P_{\ell-1}(0) - P_{\ell+1}(0) \right) = \frac{2V_0}{2\ell+1} (-1)^{k-2} \frac{(k-2)!}{(k+1)!} \left( \frac{2\ell+1}{2\ell+1} \right)^2
\]

\[
  A_\ell = V_0 \left( -1 \right)^{k-2} \frac{(k-2)!}{(k+1)!} \left( \frac{2\ell+1}{2\ell+1} \right)^2, \quad k = 2\ell + 1
\]

Hence, the solution outside is:
\[
  u_{\text{out}}(r, \theta) = \sum_{\ell=0}^{\infty} \left( -1 \right)^{\ell} \frac{(2\ell+1)!}{(2\ell+1)!} \left( \frac{r}{2\ell+1} \right)^{2\ell+1} \ P_\ell (\cos \theta)
\]

At \( r=0 \),
\[
  u_{\text{in}}(\theta) = V_0 \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{(2\ell+1)!}{(2\ell+1)!} \left( \frac{r}{2\ell+1} \right)^{2\ell+1} \ P_\ell (\cos \theta)
\]

\[
  u_{\text{out}}(r, \theta) = \frac{1}{r} \sum_{\ell=0}^{\infty} \left( -1 \right)^{\ell} \frac{(2\ell+1)!}{(2\ell+1)!} \left( \frac{r}{2\ell+1} \right)^{2\ell+1} \ P_\ell (\cos \theta)
\]