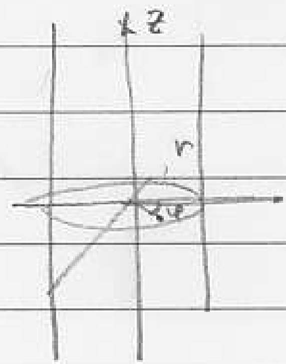
Lecture # 15
2009

Laplace equation in cylindrical coordinates



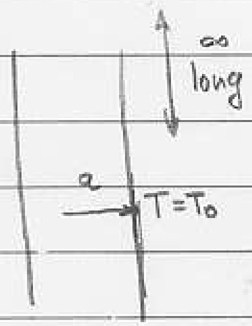
$$\nabla^2 u = 0 \qquad \text{(Boas ch 10-9)}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Separate the variables:

$$u(r,\varphi,z) = R(r)\,\Phi(\varphi)\,Z(z)$$

$$\frac{1}{Rr}\frac{d}{dr}(rR') + \frac{1}{r^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} = 0$$



① Infinite cylinder at constant temperature

no φ or z dependence! $\Phi = Z = 0$

$\frac{d}{dr}(rR') = 0 \qquad rR' = \text{const} = C_1 \qquad R = C_2 + C_1 \ln r$

Inside: solution must be limited $C_1 = 0$

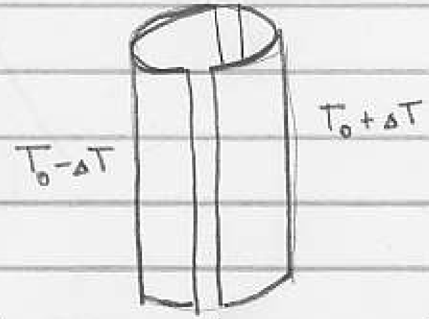
$R(r) = C_2 = T_0$

But when calculating the temperature outside we run into a problem, since $\ln r$ diverges at $r \to \infty$. There is a physical reason for that — we assumed that the cylinder is infinitely long, and at some distance away this approximation will break. To resolve this mathematically, we will add one more boundary condition; at some distance $b \gg a$ $T(r) = T_b$ (background temperature)

so for the outside region: $\begin{cases} C_2 + C_1 \ln a = T_0 \\ C_2 + C_1 \ln b = T_b \end{cases} \Rightarrow C_1 = \frac{T_0 - T_b}{\ln a/b}$, $C_2 = -C_1 \ln a + T_0$

Thus $u(r) = T_0 + C_1 \ln r/a = T_0 + \frac{T_0 - T_b}{\ln a/b} \ln r/a$

② Cylindrical "capacitor"



background temperature $-T_0$

Boundary conditions: $T(r=a) = \begin{cases} T_0+\Delta T, & 0<\varphi<\pi \\ T_0-\Delta T, & \pi<\varphi<2\pi \end{cases}$

There is still no z-dependence $(Z=0)$ but now there is φ-dependence

$$\frac{1}{rR}\frac{d}{dr}(rR') + \frac{1}{r^2}\underbrace{\left(\frac{\Phi''}{\Phi}\right)}_{=-m^2} = 0$$

$$\Phi'' = -m^2\Phi \Rightarrow \Phi = A_m\cos m\varphi + B_m \sin m\varphi$$

Since physically φ and $\varphi+2\pi$ describe the same point in space, we require

$$\Phi(\varphi) = \Phi(\varphi+2\pi) \Rightarrow m = \text{integer}$$

Equation for R: $\quad \frac{1}{Rr}\frac{d}{dr}(rR') - \frac{m^2}{r^2} = 0 \Rightarrow r\frac{d}{dr}(rR') - m^2R = 0$

solutions $\quad R = r^{\pm m}$ or $(1, \ln r)$ for $m=0$

Solution: $\quad u(r,\varphi) = C_1 + C_2\ln r + \sum_{m=1}^{\infty}(A_m\cos m\varphi + B_m\sin m\varphi)(C_m r^m + D_m \frac{1}{r^m})$

This is the general form, and now we can simplify that using the boundary conditions and requiring finite solution:

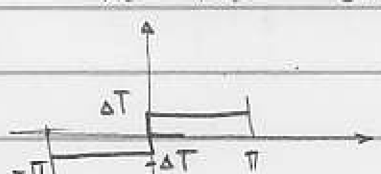
a. Inside: $C_2=0$, $D_m=0$ $\qquad$ b. Outside: $C_2=0$, $C_m=0$

c. BC are anti-symmetric in $\varphi \longrightarrow A_m=0$

d. at $r\to\infty$ $u(r,\varphi)\to T_0$, and at $r=0$ $u(0,0)=T_0 \Rightarrow C_1=T_0$

Thus $\quad u_{\text{inside}/\text{outside}}(r,\varphi) = T_0 + \sum_{m=1}^{\infty} B_m \sin m\varphi \left(\frac{r}{a}\right)^{\pm m}$

To find B_m we need to decompose the boundary function in the sine Fourie series $\quad T_{BC} = T_0 + \sum_{m=1}^{\infty} T_m \sin m\varphi$



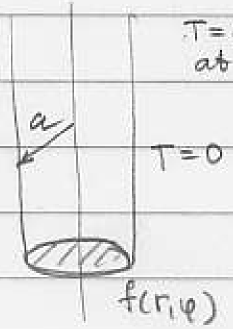
$$T_m = \frac{2}{\pi}\int_0^{\pi}\Delta T\sin m\varphi\, d\varphi = \frac{2\Delta T}{m\pi}(-\cos m\varphi)\Big|_0^{\pi} =$$

$$\begin{cases} = \dfrac{4\Delta T}{\pi m} & m=1,3,5\ldots \\ = 0 & m = 0,2,4\ldots \end{cases}$$

$$u(a,\varphi) = T_0 + \sum_{m=1}^{\infty} B_m\sin m\varphi \Rightarrow B_m = T_m = \frac{4\Delta T}{\pi m} \qquad m=1,3,5\ldots$$

$$u_{\text{inside}/\text{outside}}(r,\varphi) = T_0 + \sum_{\substack{m=1\\ \text{odd}}}^{\infty}\frac{4\Delta T}{\pi m}\sin m\varphi \left(\frac{r}{a}\right)^{\pm m}$$

③ Semi- infinite cylinder



$$\frac{1}{rR}\frac{d}{dr}(rR') + \frac{1}{r^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} = 0$$

$$\frac{\Phi''}{\Phi} = -n^2 \qquad \frac{Z''}{Z} = k^2$$

$Z'' = k^2 Z \Rightarrow Z(z) \propto e^{-kz}$ exponential

$\Phi(\varphi)$ is oscillatory

$\Phi(\varphi) = A_n \cos n\varphi + B_n \sin n\varphi$ $\quad n$ = integer

$$\frac{1}{rR}\frac{d}{dr}(rR') - \frac{n^2}{r^2} + k^2 = 0$$

$r^2R'' + rR' + (k^2r^2 - n^2)R = 0$ Bessel equ.

Solutions are: $R(r) = J_n(kr)$ for convenience $k \to k/a$

$$R(r) = J_n\left(k\frac{r}{a}\right)$$

$$u_{nk}(r,\varphi,z) = J_n\left(k\frac{r}{a}\right)(A_{nk}\cos n\varphi + B_{nk}\sin n\varphi)\, e^{-kz/a}$$

Now let's take into account the boundary conditions.

$u = 0$ at $r = a \Rightarrow J_n(k) = 0 \Rightarrow k = \alpha_i^{(n)}$ $\quad i = 1, 2 \ldots$

zeros of the Bessel function

$$u(r,\varphi,z) = \sum_{n,i} J_n\left(\alpha_i^{(n)}\frac{r}{a}\right)(A_{ni}\cos n\varphi + B_{ni}\sin n\varphi)\, e^{-\alpha_i^{(n)} z/a}$$

Then $u(r,\varphi, z=0) = f(r,\varphi) = \sum_{n,i} J_n\left(\alpha_i^{(n)}\frac{r}{a}\right)(A_n \cos n\varphi + B_n \sin n\varphi)$

So to find the coefficients A_n & B_n one has to decompose $f(r,\varphi)$ into the Fourie series in φ and Bessel series in r.

Simple example $\Rightarrow$ $\varphi(r,\varphi) = T_0$. $\quad A_n = B_n = 0$ for $n > 0$

$B_0 = 0$

$\sum_i A_i J_0\left(\alpha_i^{(0)}\frac{r}{a}\right) = T_0 \qquad \times J_0\left(\alpha_\ell^{(0)}\frac{r}{a}\right)$ and integrate

Using orthogonality condition of the Bessel functions:

$$A_\ell \cdot \frac{a^2}{2} J_1^2\left(\alpha_\ell^{(0)}\right) = T_0 \int_0^a J_0\left(\alpha_\ell^{(0)}\frac{r}{a}\right) r\,dr = T_0\, a^2 \frac{1}{\alpha_\ell^{(0)}} J_1\left(\alpha_\ell^{(0)}\right)$$

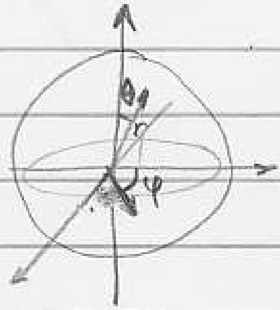
$$A_\ell = \frac{2T_0}{\alpha_\ell^{(0)} J_1\left(\alpha_\ell^{(0)}\right)}$$

$$u(r,\varphi,z) = \sum_i \frac{2T_0}{\alpha_i^{(0)} J_1\left(\alpha_i^{(0)}\right)} J_0\left(\alpha_i^{(0)}\frac{r}{a}\right) e^{-\alpha_i^{(0)} z/a}$$

Laplace equation in spherical coordinates

$$\nabla^2 u = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} = 0$$



$$u(r,\theta,\varphi) = R(r)\, P(\theta)\, \Phi(\varphi)$$

$$\frac{1}{Rr^2}\frac{d}{dr}(r^2 R') + \frac{1}{r^2 P \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\Phi''}{\Phi} = 0$$

as before: $\Phi = A_m \cos m\varphi + B_m \sin m\varphi \qquad m = 0, 1, 2 \ldots$

$$\frac{1}{Rr^2}\frac{d}{dr}(r^2 R') + \frac{1}{r^2}\underbrace{\left[\frac{1}{P}\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta P') - \frac{m^2}{\sin^2\theta}\right]}_{-\ell(\ell+1)} = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta}P + \ell(\ell+1)P = 0$$

solutions: associate Legendre functions $P_\ell^m(\cos\theta)$

Radial part : $\frac{d}{dr}(r^2 R') - \ell(\ell+1)R = 0$

$R = r^\ell$ or $r^{-\ell-1}$

"Inside" and "outside" solutions

$$u_{in}(r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} r^\ell P_\ell^m(\cos\theta)(A_{\ell m}\cos m\varphi + B_{\ell m}\sin m\varphi)$$

or $r^\ell\, Y_{\ell m}(\theta,\varphi)$

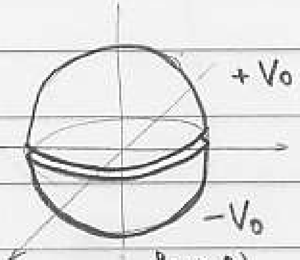
$$u_{out}(r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{1}{r^{\ell+1}} P_\ell^m(\cos\theta)(A_{\ell m}\cos m\varphi + B_{\ell m}\sin m\varphi)$$

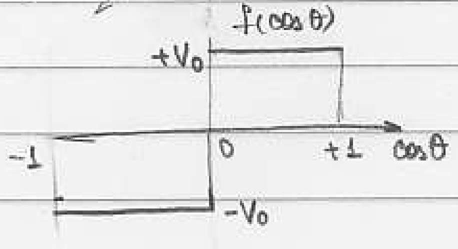
or $\frac{1}{r^{\ell+1}} Y_{\ell m}(\theta,\varphi)$

where $Y_{\ell m}(\theta,\varphi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$

spherical harmonics.

① Spherical capacitor





Boundary conditions:
In general, the values of the solution $u(r=a,\theta,\varphi) = f(\theta,\varphi)$ is given, and it has to be decomposed into the series of spherical harmonics.
In our case there is no φ-dependence on φ in our boundary conditions ($m=0$). Moreover, $f(\cos\theta)$ is odd, so we expect only odd ℓ to appear in the final summation

Solution inside: $u_{in}(r,\theta) = \sum_{\substack{\ell=1\\ odd}}^{\infty} A_\ell \left(\frac{r}{a}\right)^\ell P_\ell(\cos\theta)$

outside $u_{out}(r,\theta) = \sum_{\substack{\ell=1\\ odd}}^{\infty} A_\ell \left(\frac{a}{r}\right)^{\ell+1} P_\ell(\cos\theta)$

To find the coefficient A_ℓ, we have to decompose $f(\cos\theta)$ into the series of $P_\ell(\cos\theta)$ [or $f(x)$ into $\to P_\ell(x)$]

$$u(a,\theta) = \sum_{\substack{\ell=1\\ odd}}^{\infty} A_\ell P_\ell(x) = f(x) \qquad \times \quad P_k(x)$$

using orthogonality of P_ℓ: $A_k \frac{2}{2k+1} = \int_{-1}^{1} f(x) P_k(x)\,dx =$

$$= 2\int_0^1 V_0 P_k(x)\,dx = \{(2k+1)P_k = P'_{\ell+1} - P'_{\ell-1}\} = \frac{2V_0}{2k+1}\left(P_{k+1}(x)\Big|_0^1 - P_{k-1}(x)\Big|_0^1\right) =$$

$$= \frac{2V_0}{2k+1}\left(\underbrace{P_{k-1}(0) - P_{k+1}(0)}_{\text{both even}}\right) = \frac{2V_0}{2k+1}(-1)^{\frac{k-1}{2}}\frac{(k-2)!!}{(k+1)!!} \quad \left[\text{using } P_{2n}(0) = (-1)^n\frac{(2n-1)!!}{(2n)!!}\right.$$

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$$A_k = V_0 \, (-1)^{k-1/2} \frac{(k-2)!!}{(k+1)!!} \qquad \{k = 2\ell+1\} \Rightarrow A_{2\ell+1} = V_0(-1)^\ell \frac{(2\ell-1)!!}{(2\ell+2)!!}$$

$$u_{in}(r,\theta) = V_0 \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(2\ell-1)!!}{(2\ell+2)!!}\left(\frac{r}{a}\right)^{2\ell+1} P_\ell(\cos\theta)$$

$$u_{out}(r,\theta) = \text{———} \;\text{"}\; \text{———} \left(\frac{a}{r}\right)^{2\ell+2} \text{———} \;\text{"}\; \text{———}$$