General Sturm-Liouville problem
\[ \frac{d}{dx} \left( f(x)y' \right) - q(x)y + \lambda w(x)y = 0 \]

Solutions: complete set of orthogonal functions \( y_\alpha \)
\[ \int y_\alpha \cdot y_\beta \cdot w(x) \, dx = 0 \quad \text{for} \ x \neq x' \]

In particular, if the equation can be written as
\[ \frac{d}{dx} \left[ d(x)w(x)y' \right] + \lambda w(x)y = 0 \]
then the solutions are orthogonal polynomials, that can be calculated using Rodrigues formula:
\[ y_n(x) = N \frac{1}{w(x)} \frac{d^n}{dx^n} \left( w(x) [d(x)]^n \right) \]

Also, many polynomials can be reproduced using a corresponding generation function:
\[ \Phi(x, h) = \sum_{n=0}^{\infty} y_n(x) h^n \]

Example of the polynomials:

Hermite polynomials: \( y'' - 2xy' + 2ny = 0 \) \( \Rightarrow \) \( \frac{d}{dx} \left( e^{-x^2} y' \right) + 2ne^{-x^2} y \)
here \( d(x) = 1 \), \( w(x) = e^{-x^2} \)
Rodrigues formula: \( H_n = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \)
Generation function: \( \Phi(x, h) = e^{2xh - h^2} = \sum_{n=0}^{\infty} H_n(x) \frac{h^n}{n!} \)
Orthogonality: \( \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \, dx = \sqrt{\pi} 2^n n! \delta_{nm} \)

Laguerre polynomials: \( xy'' + (1-x)y' + ny = 0 \) \( \Rightarrow \) \( \frac{d}{dx} \left[ xe^{-x} y' \right] + ne^{-x} y = 0 \)
here \( d(x) = x \), \( w(x) = e^{-x} \): Rodrigues formula: \( L_n(x) = \frac{1}{n!} e^{x} \frac{d^n}{dx^n} (xe^{-x}) \)
Generation function: \( \Phi(x, h) = \frac{e^{-1-h} h^n}{1-h} = \sum_{n=0}^{\infty} L_n(x) h^n \)
Orthogonality: \( \int_{0}^{\infty} e^{-x} L_n(x) L_m(x) \, dx = \delta_{nm} \)
Bessel functions:

\[ x^2 y'' + x y' + (x^2 - p^2) y = 0 \implies \frac{1}{x} \frac{d}{dx} \left( x y' \right) + (1 - \frac{p^2}{x^2}) y = 0 \]

Solutions \( J_p(x) \) → set of orthogonal functions.

Associate Legendre polynomials:

\[ (1-x^2) y'' - 2 x y' + \left( l(l+1) - \frac{m^2}{1-x^2} \right) y = 0 \implies \frac{d}{dx} \left( (1-x^2) y' \right) + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] y = 0 \]

or, using \( x = \cos \theta \)

\[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta y' \right] + \left[ l(l+1) - \frac{m^2}{\sin \theta} \right] y = 0 \]

Solutions: \( P_e^m(x) \) or \( P_l^m(\theta) \) → associate Legendre functions (not all \( P_e^m(x) \) are polynomials).

\[ P_e^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_e(x) \]

but this expression works only for positive \( m \).

In general, we will have to use Rodrigues formula for \( P_e^m(x) \):

\[ P_e^m(x) = (1-x^2)^{m/2} \frac{1}{2^m m!} \frac{d^m}{dx^m} (x^2-1)^m \]

Because of the definition \( P_e^m \) and \( P_l^m \),

\[ P_e^m(x) = (-1)^m \frac{(l+m)!}{(l-m)!} P_l^{-m}(x) \]

Orthogonality: associate Legendre functions of the same argument \( m \) are orthogonal:

\[ \int P_e^m(x) P_l^m(x) \, dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \]

See!
Examples: let's calculate a few first ass. Legendre functions:

\[ P_0(x) = P_0'(x) = 1 \]
\[ P_1(x) = \frac{1}{2} x, \quad P_1'(x) = \frac{1}{2} (1 - x^2) \]

Two "special cases"

For \( m = l \):
\[ P_l = (1-x^2)^{l/2} \frac{1}{2^l \cdot l!} \frac{d^l}{dx^l} (x^2-1)^l = (1-x^2)^{l/2} \frac{(2l)!}{2^l \cdot l!} \]

For \( m = -l \):
\[ P_{-l} = (1-x^2)^{l/2} \frac{1}{2^l \cdot l!} (-1)^l (x^2-1)^l = (1-x^2)^{l/2} \frac{(-1)^l}{2^l \cdot l!} \]

Associate Legendre functions are essential components of spherical functions (or spherical harmonics)

Many major physics equations involve \( \nabla^2 F = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) F \)

In spherical coordinates, this operator is:
\[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \left( \frac{1}{\sin^2 \theta} \frac{d}{d\theta} \sin \theta \frac{d}{d\theta} + \frac{1}{\sin^2 \theta} \frac{d^2}{d\phi^2} \right) F = 0 \]

If one solves an equation \( \nabla^2 F = 0 \) or \( \nabla^2 F = f(r) \) for all angular-dependent terms are inside \( \Phi \)?

So we can separate the variables and search for \( F(r, \theta, \phi) = R(r) Y(\theta, \phi) \)

Then (for the sake of argument) I will use \( \nabla^2 F = 0 - \text{Laplace eq} \)

\[ \frac{1}{r^2} Y(\theta, \phi) \left[ R'' + \frac{2}{r} R' \right] + \frac{R}{r^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y = 0 \]

\[ = \text{const} = -l(l+1) \]
Then the equation for the spherical part
\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} + \ell (\ell + 1) Y = 0
\]

If we separate variables again: \( Y(\theta, \phi) = P(\theta) \Phi(\phi) \)
\[
\Phi \frac{d}{d\theta} \left[ \sin \theta \Phi'(\theta) \right] + \frac{P(\theta)}{\sin \theta} \Phi''(\theta) + \ell (\ell + 1) \Phi(\phi) = 0
\]
\[
\frac{1}{P} \frac{d}{d\theta} \left[ \sin \theta P(\theta) \right] + \frac{1}{\sin^2 \theta} \left[ \frac{\Phi''}{\Phi} \right] + \ell (\ell + 1) = 0
\]

Azimuthal part: \( \Phi'' = -m^2 \Phi \implies \Phi = e^{im\phi} \)
Can we say anything about values of \( m \)?
In physical world, \( \phi, \phi + 2\pi, \phi + 4\pi \) correspond to the same point in space, so it is reasonable to request that \( \Phi(\phi) = \Phi(\phi + 2\pi) \implies e^{im\phi} = e^{im(\phi + 2\pi)} \)

\( \Rightarrow m \) should be integer

Back to the equation for \( \theta \):
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta P(\theta) \right] - \frac{m^2}{\sin^2 \theta} P(\theta) + \ell (\ell + 1) P(\theta) = 0
\]
solutions \( \Rightarrow \) associate Legendre functions \( P^m(\theta) \)
Thus: \( Y(\theta, \phi) = Y_{\ell m}(\theta, \phi) = A_{\ell m} P^m(\theta) \Phi_{m}(\phi) \)

Orthogonality: we want to request that \( Y_{\ell m}(\theta, \phi) \) are orthonormal:
\[
\int_0^{2\pi} \int_0^\pi Y_{\ell m} \cdot Y_{\ell ' m'} \sin \theta \, d\theta \, d\phi = \delta_{\ell \ell'} \delta_{mm'}
\]

That means that:
\[
A_{\ell m} A_{\ell ' m'} \left( \int_0^{2\pi} e^{im\phi} e^{-im'\phi} \, d\phi \right) \left( \int_0^\pi P^m P^{m'} \sin \theta \, d\theta \right) = \delta_{\ell \ell'} \delta_{mm'}
\]
\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i(m-m')\phi} \, d\phi = \delta_{mm'}
\]
\[
\frac{1}{2\pi} \int_0^\pi e^{i(m-m')\phi} \sin \theta \, d\theta = \delta_{mm'}
\]
\[
x = \cos \theta
\]
\[
2\pi A_{\ell m} A_{\ell ' m'} \int_0^\pi P^{m}(x) P^{m'}(x) \, dx = \delta_{\ell \ell'} \delta_{mm'}
\]
\[ 2\pi |A_m|^2 \delta_{m'm} = \frac{2}{(2l+1)} \frac{(l+m)!}{(l-m)!} \delta_{lm} = \delta_{m'm} \delta_{l'l} \]

\[ \Rightarrow A_m = -\sqrt{\frac{2l+1}{4\pi}} \frac{(l+m)!}{(l-m)!} \cdot (-1)^m \]

Thus: Spherical harmonics \( Y_{lm} = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{(2\pi)(l+m)!}} P_l^m(\cos \theta) e^{i\phi} \)

First few spherical functions:

\( l=m=0 \) \( Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \) (\( l=0, m=0 \))

\( l=1, m=0 \) \( Y_{10} = -\sqrt{\frac{3}{4\pi}} \cos \theta \)

\( l=1, m=1 \) \( Y_{11} = (-1)^* \sqrt{\frac{3}{4\pi}} \frac{1}{2} \sin \theta \sin \phi e^{i\phi} = -\sqrt{\frac{3}{8\pi}} \sin \theta \sin \phi e^{i\phi} \)

\( l=1, m=-1 \) \( Y_{1-1} = (-1)^* \sqrt{\frac{3}{4\pi}} \frac{1}{2} \sin \theta \sin \phi \) \( e^{i\phi} = \sqrt{\frac{3}{8\pi}} \sin \theta \sin \phi e^{i\phi} \)

Physical meaning

Spherical harmonics are the eigenvalues of the angular momentum operator in QM

\( \mathbf{\vec{l}} = \mathbf{\vec{r}} \times \mathbf{\vec{p}} \) (in classical physics)

In quantum mechanics \( \mathbf{\hat{l}} \) becomes an operator: \( \mathbf{\hat{l}} = -i\hbar \left( \mathbf{\vec{r}} \times \mathbf{\vec{p}} \right) \)

\( \mathbf{\hat{l}}_x, \mathbf{\hat{l}}_y, \mathbf{\hat{l}}_z \) do not commute (i.e., they cannot be measured simultaneously). Important operators are:

\( \mathbf{\hat{l}}^2 = (\mathbf{\vec{l}} \cdot \mathbf{\vec{l}}) \), and \( \mathbf{\hat{l}}_z = -\frac{\hbar}{2i} \frac{\partial}{\partial \phi} \)

In spherical coordinates

\( \mathbf{\hat{l}}^2 = -\frac{\hbar^2}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{\hbar^2}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \)

and

\( \mathbf{\hat{l}}_z = -i\hbar \frac{2}{\sin \theta} \)
It is easy to see that $\hat{L}_z \Phi_m(z) = -i\hbar \frac{\partial}{\partial z} \left( \frac{1}{\sqrt{2\pi}} e^{imz} \right) = \frac{\hbar}{2\pi} \frac{\partial}{\partial z} e^{imz} = \hbar m \Phi_m(z)$

and $Y_{em}(\theta, \phi)$ are the eigen functions of $\hat{L}_z^2$

$\hat{L}_z^2 Y_{em}(\theta, \phi) = -\hbar^2 \left( \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{\hbar^2}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) Y_{em}(\theta, \phi) = \hbar^2 \ell (\ell + 1) Y_{em}(\theta, \phi)$

Thus, if the state of the system is described by a particular spherical harmonics $Y_{em}(\theta, \phi)$
then its total angular momentum $\langle \hat{L}_z^2 \rangle = \hbar^2 \ell (\ell + 1)$,

and its z-component $\langle \hat{L}_z \rangle = m$