Legendre polynomials (continued)

Equation for LP: $\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \ell(\ell+1)y = 0$ (1)

or

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0 \qquad (2)$$

One more form of Legendre equation

$x \to \cos\theta$; $1-x^2 = \sin^2\theta$, $dx = -\sin\theta\, d\theta$

Using the self-adjourn (the top) form of the equation:

$$\frac{1}{-\sin\theta}\frac{d}{d\theta}\left[\sin^2\theta \frac{1}{-\sin\theta}\frac{d}{d\theta}y\right] + \ell(\ell+1)y = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dy}{d\theta}\right] + \ell(\ell+1)y = 0 \qquad (3)$$

Legendre polynomials are solutions of this equation for integer ℓ, finite at $-1 \le x \le 1$. For each ℓ $P_\ell(x)$ is the polynomial of order ℓ, the normalization is set such that $P_\ell(1) = 1$.

LP have parity: $P_\ell(x) = (-1)^\ell P_\ell(-x)$

Rodrigez formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{\partial^\ell}{\partial x^\ell}(x^2-1)^\ell$$

First few polynomials:

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = -\frac{1}{2} + \frac{3}{2}x^2$

$P_3(x) = -\frac{3}{2}x + \frac{5}{2}x^3$

$$(1+\alpha)^{-1/2} = 1 - \tfrac{1}{2}\alpha + \tfrac{3}{8}\alpha^2 - \dots$$

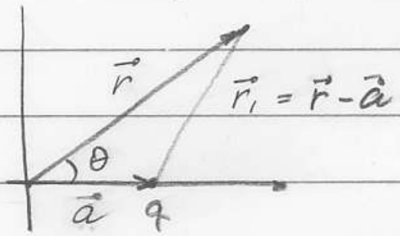
Generation function

$$\Phi(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} = \underbrace{(1 \overbrace{-2xh+h^2}^{\alpha})^{-1/2}}_{\text{treat } x \text{ as a parameter and decompose into a Taylor series with respect to "}h\text{"}} =$$

$$= 1 - \frac{1}{2}(-2xh+h^2) + \frac{3}{8}(-2xh+h^2)^2 - \dots = \underset{P_0(x)}{1} + \underset{P_1(x)}{x}h + \underset{P_2(x)}{\left(-\frac{1}{2}+\frac{3}{2}x^2\right)}h^2 + \dots$$

$$\Phi(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} = \sum_{\ell=0}^{\infty} P_\ell(x)\, h^\ell$$

Interesting physical meening



$$\varphi = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{r_1}$$

$$r_1 = |\vec{r}-\vec{a}| = \sqrt{r^2+a^2-2ra\cos\theta} =$$

$$= r\sqrt{1+\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta}$$

$$\varphi = \frac{q}{2\pi\varepsilon_0}\,\frac{1}{r}\,\frac{1}{\sqrt{1+\left(\frac{q}{r}\right)^2-2\left(\frac{q}{r}\right)\cos\theta}} = \underbrace{\frac{q}{2\pi\varepsilon_0}\,\frac{1}{r}\sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(\frac{q}{r}\right)^\ell}_{\text{multipole expansion}}$$

The generation function is useful to derive the recurrence relations

$$\Phi(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} = \sum_{\ell=0}^{\infty} P_\ell(x)\, h^\ell$$

There are two basic recurrence relations: one uses $\frac{\partial\Phi}{\partial x}$, and the other $\frac{\partial\Phi}{\partial h}$

 $\Phi(x,h)$

① $$\frac{\partial\Phi}{\partial h} = \frac{x-h}{(1-2xh+h^2)}\frac{1}{\sqrt{1-2xh+h^2}} = \frac{x-h}{(1-2xh+h^2)}\sum_{\ell=0}^{\infty} P_\ell(x)h^\ell$$

$$\frac{\partial\Phi}{\partial h} = \sum_{\ell=0}^{\infty} \ell P_\ell(x)x^{\ell-1}$$

$$(x-h)\sum_{\ell=0}^{\infty} P_\ell(x)h^\ell = (1-2xh+h^2)\sum_{\ell=0}^{\infty} \ell P_\ell(x)h^{\ell-1}$$

$$\sum_{\ell=0}^{\infty} xP_\ell(x)h^\ell - \sum_{\ell=0}^{\infty} P_\ell(x)h^{\ell+1} = \sum_{\ell=0}^{\infty} \ell P_\ell(x)h^{\ell-1} - \sum_{\ell=0}^{\infty} 2\ell x P_\ell(x)h^\ell + \sum_{\ell=0}^{\infty} \ell P_\ell(x)h^{\ell+1}$$

Now we make sure the equality stands for each "h^ℓ" term:

h^ℓ: $xP_\ell(x) - P_{\ell-1}(x) = (\ell+1)P_{\ell+1}(x) - 2\ell x P_\ell(x) + (\ell-1)P_{\ell-1}(x)$

$$\boxed{(\ell+1)P_{\ell+1}(x) - (2\ell+1)xP_\ell(x) + \ell P_{\ell-1}(x) = 0}$$

Usually written for $\ell, \ell-1, \ell-2$ terms

$$\boxed{\ell P_\ell(x) - (2\ell-1)xP_{\ell-1}(x) + (\ell-1)P_{\ell-2}(x) = 0}$$

② $$\frac{\partial\Phi}{\partial x} = \frac{h}{(1-2xh+h^2)}\frac{1}{\sqrt{1-2xh+h^2}} = \frac{h}{1-2xh+h^2}\cdot\sum_{\ell=0}^{\infty} P_\ell(x)h^\ell$$

$$\frac{\partial\Phi}{\partial x} = \sum_{\ell=0}^{\infty} P_\ell'(x)h^\ell$$

$$(1-2xh+h^2)\sum_{\ell=0}^{\infty} P_\ell'(x)h^\ell = h\sum_{\ell=0}^{\infty} P_\ell(x)h^\ell$$

$$\sum_{\ell=0}^{\infty} P_\ell'(x)h^\ell - \sum_{\ell=0}^{\infty} 2xP_\ell'(x)h^{\ell+1} + \sum_{\ell=0}^{\infty} P_\ell'(x)h^{\ell+2} = \sum_{\ell=0}^{\infty} P_\ell(x)h^{\ell+1}$$

$h^{\ell+1}$: $\underline{P_{\ell+1}' - 2xP_\ell'(x) + P_{\ell-1}'(x) = P_\ell(x)}$

Other recurrence relations (you'll derive them as a part of the homework)

(3) $x P_\ell'(x) - P_{\ell-1}'(x) = \ell P_\ell(x)$

We can use (3) to simplify (2):

$$2x P_\ell'(x) = 2P_{\ell-1}' + 2\ell P_\ell(x) \quad \Rightarrow (2)$$

$$P_{\ell+1}' - 2P_{\ell-1}' - 2\ell P_\ell + P_{\ell-1}' = P_\ell(x)$$

(2') $$P_{\ell+1}' - P_{\ell-1}' = (2\ell+1) P_\ell$$

(4) $(1-x^2) P_\ell'(x) = \ell P_{\ell-1}(x) - \ell x P_\ell(x)$

(5) $(1-x^2) P_{\ell-1}'(x) = \ell x P_{\ell-1}(x) - \ell P_\ell(x)$

(6) $P_\ell'(x) - P_{\ell-1}'(x) = \ell P_\ell(x)$

Orthogonality of LP: We preoved it in general for any Sturm-Liouville problem solutions

Need to proof: $\int_{-1}^{1} P_\ell(x) P_k(x)\,dx = 0$ for $\ell \neq k$

$$P_k \times \frac{d}{dx}\left[(1-x^2) P_\ell'\right] + \ell(\ell+1) P_\ell = 0$$
$$- \; P_\ell \times \frac{d}{dx}\left[(1-x^2) P_k'\right] + k(k+1) P_k = 0$$

subtract and integrate $\int_{-1}^{1} \dots dx$

< side note: $\frac{d}{dx}\left[(1-x^2) P_k P_\ell'\right] = P_k \frac{d}{dx}\left[(1-x^2) P_\ell'\right] + (1-x^2) P_k' P_\ell'$ >

$$\underbrace{\int_{-1}^{1} \frac{d}{dx}\left[(1-x^2)(P_k P_\ell' - P_\ell P_k')\right] dx}_{\cancel{(1-x^2)(P_k P_\ell' - P_\ell P_k')\Big|_{-1}^{1}} = 0} + \left\{\ell(\ell+1) - k(k+1)\right\} \int_{-1}^{1} P_\ell P_k\,dx = 0$$

$$\Rightarrow \int_{-1}^{1} P_\ell P_k\,dx = 0 \quad \text{for } \ell \neq k$$

Let's calculate normalization: $\int_{-1}^{1} P_\ell^2(x)dx$

$$\ell P_\ell(x) = x P_\ell'(x) - P_{\ell-1}'(x)$$

$$\int_{-1}^{1} \ell P_\ell^2(x) = \int_{-1}^{1} x P_\ell' \cdot P_\ell dx - \underbrace{\int_{-1}^{1} P_\ell \cdot P_{\ell-1}'(x)\, dx}_{=0} =$$

due to orthogonality

$$= \frac{1}{2}\int_{-1}^{1} x\, d\left[P_\ell(x)\right]^2 - \int_{-1}^{1} P_\ell \cdot \underbrace{\left[\text{polynomial of order } \ell-1\right]}_{= a_1 P_{\ell-2} + a_2 P_{\ell-3} + \dots} dx =$$

$$= \underbrace{\frac{1}{2} x P_\ell^2(x)\Big|_{-1}^{1}}_{=1} - \frac{1}{2}\int_{-1}^{1} P_\ell^2(x)dx$$

$$\ell \cdot \int_{-1}^{1} P_\ell^2(x)dx = 1 - \frac{1}{2}\int_{-1}^{1} P_\ell^2(x)\,dx$$

$$\int_{-1}^{1} P_\ell^2(x)dx = \frac{2}{2\ell+1}$$

And we can write down the orthogonality conditions:

$$\boxed{\int_{-1}^{1} P_\ell(x)\, P_k(x)\, dx = \frac{2}{2\ell+1}\,\delta_{\ell k}}$$

Decomposition of a function into a series of LP

$$f(x) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x)$$

$$\int_{-1}^{1} f(x)\, P_k(x)\, dx = \sum_{\ell=0}^{\infty} c_\ell \int_{-1}^{1} P_\ell(x)\, P_k(x) dx = \frac{2}{2k+1}\, c_k$$

$$\boxed{c_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)\, P_k(x)\, dx}$$

Examples: $P_3'(x) = -\frac{3}{2} + \frac{15}{2}x^2$ $\Rightarrow$ will use $P_0 = 1,\ P_1 = x,\ P_2 = \frac{3}{2}x^2 - \frac{1}{2}$

a) $P_3'(x) = 5\cdot\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + \frac{5}{2} - \frac{3}{2} = 5P_2(x) + 1.$

b) $P_3'(x) = \sum_{\ell=0}^{2} c_\ell P_\ell(x)$ $\quad c_0 = \frac{1}{2}\int_{-1}^{1}\left(-\frac{3}{2} + \frac{15}{2}x^2\right)dx = \frac{1}{4}\left(-3x + 5x^3\right)\Big|_{-1}^{1} = 1$

$c_1 = 0$ (due to parity)

$$c_2 = \frac{5}{2}\int_{-1}^{1}\left(-\frac{3}{2} + \frac{15}{2}x^2\right)\left(\frac{3}{2}x^2 - \frac{1}{2}\right)dx = \frac{15}{8}\int_{-1}^{1}(15x^4 - 8x^2 + 1)\,dx =$$

$$= \frac{15}{8}\left(3x^5 - \frac{8}{3}x^3 + x\right)\Big|_{-1}^{1} = \frac{15}{8}\left(6 - \frac{16}{3} + 1\right)\Big|_{-1}^{1} = 5$$

c) Use recurrence relation $P_{\ell+1}' - P_{\ell-1}' = (2\ell+1)P_\ell$

$$P_3' - \underbrace{P_1'}_{=1} = 5P_2 \quad\Rightarrow\quad \underline{P_3'(x) = 5P_2'(x) + 1.}$$