Extra-credit problem – Kramers-Kronig relations

In optics Kramers-Kronig relations provide a relation between the real and imaginary parts of the linear susceptibility $\chi(\omega)$ of a substance for an electromagnetic wave with frequency $\omega$.

$$\text{Re}(\chi(\omega)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\chi'(\omega')d\omega'}{\omega'-\omega}$$

$$\text{Im}(\chi(\omega)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\chi'(\omega')d\omega'}{\omega'-\omega}$$

These are very general expressions valid for majority of optical media. They only assume that $\chi(\omega)$ is analytic and finite in a upper half plane.

Prove the Kramers-Kronig relations, using the residue theorem, using the following integral:

$$\text{Int} = \int_{-\infty}^{\infty} \frac{\chi'(\omega')d\omega'}{\omega'-\omega},$$

and then turning in into a contour integral in the upper half plane.

*Optical susceptibility $\chi(\omega)$ characterizes the absorption and the refractive index of the optical medium. In particular, the absorption per unit length is proportional to $\text{Im}(\chi(\omega))$, and the refractive index is $n=1+4\pi\text{Re}(\chi(\omega))$. 