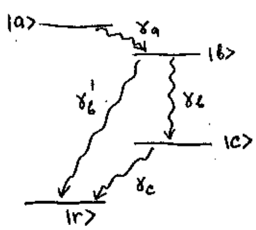
PHYS 404/690

Midterm test *(March 27)*

This is an open book test, you can use the textbook and your class notes. This is ok to use your computer if you are using the electronic textbook version, but you are not allowed to search for the solutions on-line or to communicate with other people.
The questions marked with $*$ are required for graduate students only.

Problem 1 (35/40* points)



Consider a four-level system shown. The reservoir level r is the ground-state. Upper level a decays into level b with the decay rate γ_a, level b decays into the level c with the rate γ_b, and to the reservoir level r with rate γ_b'. Finally, the lower level c decays into the ground state with the rate γ_c. An electric discharge effectively repopulates the state a at a rate λ.
(a) Write down the equations for the steady-state **populations** of the levels b and c in the absence of any external optical fields.
(b) Assuming the atomic populations are fixed to the values you've calculated above, find the expression for susceptibility $\chi(\Delta)$ for an optical field, nearly resonant with the $|b\rangle \to |c\rangle$ transition. Here Δ is the frequency difference between the optical field and the transition frequency.
*(c)** Show that depending on the values of decays and the repumping rate, it is possible to realize either absorption or amplification for the optical field. Find the conditions necessary to achieve amplification.

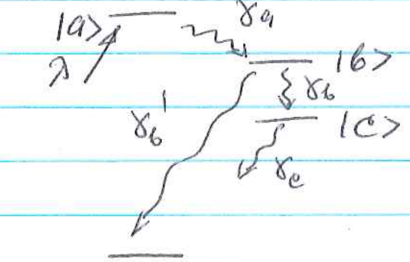
Problem 2 (35/40* points)
In class we discuss the potential advantages of using "NOON" states for quantum enhanced measurements. Suppose you are tasked with a charge of producing such a state with $N = 3$ photons, shared between two output of a beam splitter. You have two options: send all three identical photons on one input of the beam splitter or send two photons in one input port, and the third one in the second input port. In either case, the desired state (i.e. all three photons emerging in one of the output) will be generated only in fraction of the trials.
(a) Calculate the probability to detect a NOON state $\left(|3\rangle|0\rangle + e^{i\phi}|0\rangle|3\rangle\right)/\sqrt{2}$ in both cases. Determine the value of the phase ϕ. In which case the rate of success is higher?
*(b)** Suppose that you have an unlimited number of single-photon sources and beam splitters. Construct a possible experimental arrangement to realize the $N = 3$ NOON state discussed above, indicating what kind of state selection is necessary in which stage.

Problem 3 (30/20* points)
Earlier in class we have discussed the classical second-order coherence function $\gamma^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle}$. The quantum analog of this function for a single-mode quantum field is $\gamma^{(2)} = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}\rangle}{\langle \hat{a}^\dagger \hat{a}\rangle^2}$.
Calculate the values of $\gamma^{(2)}$ for a number state $|n\rangle$ and a coherent state $|\alpha\rangle$.

Midterm solutions

Problem 1

a)



$$\dot\rho_{aa} = \lambda - \gamma_a \rho_{aa}$$

$$\dot\rho_{bb} = \gamma_a \rho_{aa} - (\gamma_b + \gamma_b')\rho_{bb}$$

$$\dot\rho_{cc} = \gamma_b \rho_{bb} - \gamma_c \rho_{cc}$$

Steady-state : $\dot\rho_{ii} = 0$

$$\rho_{aa} = \lambda / \gamma_a$$

$$\rho_{bb} = \frac{\lambda}{\gamma_b + \gamma_b'} \qquad \rho_{cc} = \frac{\gamma_b}{\gamma_b + \gamma_b'}\frac{\lambda}{\gamma_c}$$

b) $$\dot\rho_{bc} = -(\gamma_{bc} + i\Delta)\rho_{bc} + i\frac{\wp_{bc}\mathcal{E}}{2\hbar}(\rho_{bb} - \rho_{cc})$$

$$\dot\rho_{bc} = 0$$

$$\rho_{bc} = i\frac{\wp_{bc}\mathcal{E}}{2\hbar}\frac{\rho_{bb} - \rho_{cc}}{\gamma_{bc} + i\Delta}$$

$$\chi(\Delta) = \frac{N\wp_{bc}\rho_{bc}}{\varepsilon_0 (\mathcal{E}/2)} = i\frac{\wp_{bc}^2}{\hbar\varepsilon_0}N\frac{(\rho_{bb} - \rho_{cc})}{\gamma_{bc} + i\Delta}$$

$$\rho_{bb} - \rho_{cc} = \frac{\lambda}{\gamma_b + \gamma_b'}\left(1 - \frac{\gamma_b}{\gamma_c}\right) \qquad \gamma_{bc} = \frac{\gamma_b + \gamma_b' + \gamma_c}{2}$$

c) Absorption / gain coefficient $\alpha(\Delta) = \frac{k}{2}\chi''$ (imaginary part)

$$\alpha(\Delta) = \frac{\wp_{bc}^2 k}{2\hbar\varepsilon_0}N(\rho_{bb} - \rho_{cc})\frac{\gamma_{bc}}{\gamma_{bc}^2 + \Delta^2}$$

Absorption $\alpha(\Delta) < 0 \Rightarrow \rho_{bb} - \rho_{cc} < 0$

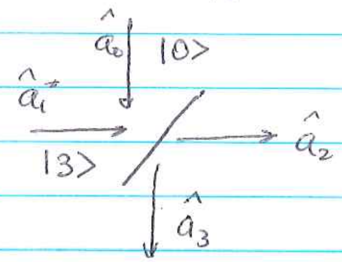
Amplification $\alpha(\Delta) > 0 \Rightarrow \rho_{bb} - \rho_{cc} > 0$

Conditions for amplification

$\gamma_b < \gamma_c$ for any value of λ.

Problem 2

a) $|3\rangle = \frac{1}{\sqrt{6}} \hat{a}^{+3}|0\rangle \qquad |2\rangle = \frac{1}{\sqrt{2}}\hat{a}^{+2}|0\rangle \qquad |1\rangle = \hat{a}^{+}|0\rangle$



$$\hat{a}_0^+ = (i\hat{a}_2^+ + \hat{a}_3^+)$$
$$\hat{a}_1^+ = (\hat{a}_2^+ + i\hat{a}_3^+)$$

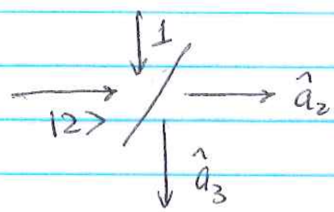
$$|3\rangle_1|0\rangle_0 = \frac{1}{\sqrt{6}}(\hat{a}_1^+)^3|0\rangle_1|0\rangle_0 \Rightarrow \frac{1}{4\sqrt{3}}(\hat{a}_2^+ + i\hat{a}_3^+)^3\ |0\rangle_2|0\rangle_3$$

$$= \frac{1}{4\sqrt{3}}\left((\hat{a}_2^+)^3 - i(\hat{a}_3^+)^3 + 3i(\hat{a}_2^+)^2\hat{a}_3^+ - 3\hat{a}_2^+(\hat{a}_3^+)^2\right)|0\rangle_2|0\rangle_3 =$$

$$= \frac{1}{4\sqrt{3}}\left[\sqrt{6}\left(|3\rangle_2|0\rangle_3 - i|0\rangle_2|3\rangle_3\right) + 3\sqrt{2}\left(i|2\rangle_2|1\rangle_3 - |1\rangle_2|2\rangle_3\right)\right]$$

$$= \frac{1}{2}\cdot\underbrace{\left[\frac{|3\rangle_2|0\rangle_3 - i|0\rangle_2|3\rangle_3}{\sqrt{2}}\right]}_{\text{NOON state}\quad \varphi = -\pi/2} + \frac{\sqrt{6}}{4}\left[i|2\rangle_2|1\rangle_3 - |1\rangle_2|2\rangle_3\right]$$

Probability of the NOON state generation $P_{NOON} = \frac{1}{4}$



$$|2\rangle_1|1\rangle_0 = \frac{1}{\sqrt{2}}(\hat{a}_1^+)^2\hat{a}_0^+\ |0\rangle_1|0\rangle_0 \Rightarrow$$

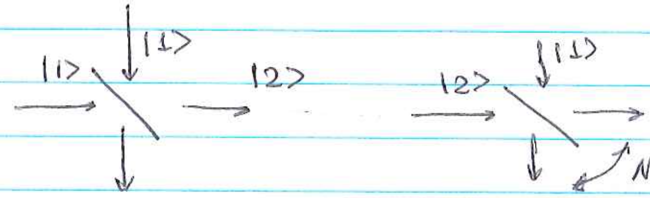
$$= \frac{1}{4}\left(\hat{a}_2^+ + i\hat{a}_3^+\right)^2\left(i\hat{a}_2^+ + \hat{a}_3^+\right)|0\rangle_2|0\rangle_3 =$$

$$= \frac{1}{4}\left[i(\hat{a}_2^+)^3 - (\hat{a}_3^+)^3 - \hat{a}_2^{+2}\hat{a}_3^+ + i\hat{a}_2^+\hat{a}_3^{+2}\right]|0\rangle_2|0\rangle_3 =$$

$$= \frac{1}{4}\left[\sqrt{6}\,i\left[|3\rangle_2|0\rangle_3 + i|0\rangle_2|3\rangle_3\right] + \sqrt{2}\left(i|1\rangle_2|2\rangle_3 - |2\rangle_2|1\rangle_3\right)\right]$$

$$= \frac{\sqrt{3}}{2}i\underbrace{\left[\frac{|3\rangle_2|0\rangle_3 + i|0\rangle_2|3\rangle_3}{\sqrt{2}}\right]}_{\text{NOON state}\quad \varphi = \pi/2} - \frac{1}{2\sqrt{2}}\left[|2\rangle_2|1\rangle_3 - i|1\rangle_2|2\rangle_3\right]$$

$P_{NOON} = 3/4 \leftarrow$ much more efficient b)



NOON state in 75% cases

post-select the cases when no photons in this port

Problem 3

$$g^{(2)} = \frac{\langle \hat{a}^{+}\hat{a}^{+}\hat{a}\hat{a}\rangle}{\langle \hat{a}^{+}\hat{a}\rangle^{2}}$$

Number state: $\langle n|\hat{a}^{+}\hat{a}|n\rangle = n$

$$\langle n|\hat{a}^{+}\hat{a}^{+}\hat{a}\hat{a}|n\rangle = n(n-1)$$

$$g^{(2)}_{2} = \frac{n(n-1)}{n^2} = 1 - \frac{1}{n}$$

Coherent state $\langle \alpha|\hat{a}^{+}\hat{a}|\alpha\rangle = |\alpha|^2$

$$\langle \alpha|\hat{a}^{+}\hat{a}^{+}\hat{a}\hat{a}|\alpha\rangle = |\alpha|^4$$

$$g^{(2)} = \frac{|\alpha|^4}{|\alpha|^4} = 1$$ same as for a classical field.