Parametric processes

$\vec{P} = \chi^{(1)} \vec{E}$ linear susceptipility

Non-linear effects

$$\vec{P} = \chi^{(1)} \vec{E} + \chi^{(2)} E^2 + \chi^{(3)} E^3 + \dots$$

in general χ may be tensors

$\chi^{(2)}$ - second-order nonlinear susceptibility

$\chi^{(3)}$ - third-order ——— " ———

Second-order nonlinear polarization

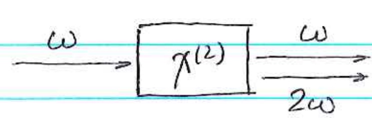
$$P^{(2)} = \chi^{(2)} E^2 \qquad E = \frac{1}{2} E_0 e^{-i\omega t} + \frac{1}{2} E_0^* e^{i\omega t}$$

$$P^{(2)}(t) = \frac{1}{4} \chi^{(2)} \left\{ E_0^2 e^{-2\omega t} + (E_0^*)^2 e^{i\omega t} + 2|E_0|^2 \right\}$$

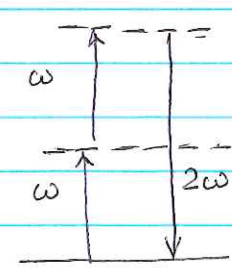
Two contribution: zero frequency (DC term) $\rightarrow$ optical rectification, creates static electric field

second harmonics (2ω) generation $\rightarrow$ results in the generation of an optical field on the doubled frequency

$$\nabla^2 E_{2\omega} - \frac{n^2}{c^2} \frac{\partial^2 E_{2\omega}}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial^2 P^{(2)}}{\partial t^2}$$



or

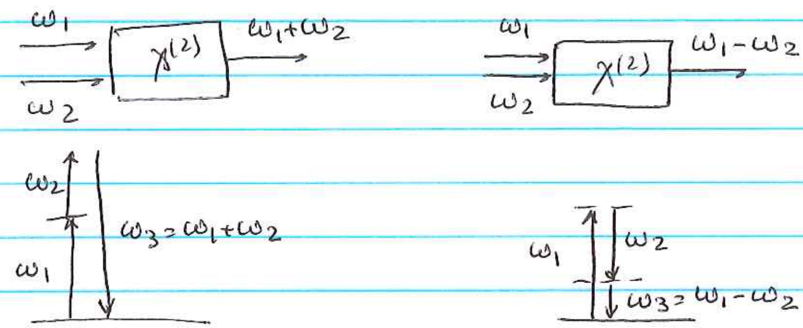


Sum- and difference frequency generation

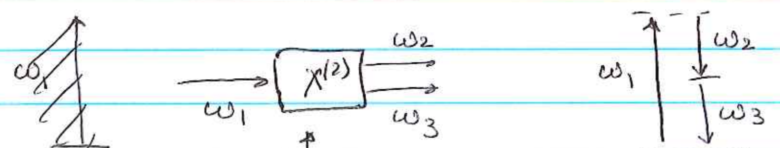
$$E(t) = E_1(t) + E_2(t) = E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + c.c.$$

$$P^{(2)} = \chi^{(2)} E^2 = \underbrace{2\chi^{(2)}\left(|E_1|^2 + |E_2|^2\right)}_{\text{optical rectification}} + \underbrace{\chi^{(2)}\left(E_1^2 e^{-i2\omega_1 t} + E_2^2 e^{-i2\omega_2 t} + c.c\right)}_{\text{SHG}}$$

$$+ 2\chi^{(2)}\left[\underbrace{E_1 E_1 e^{-i(\omega_1+\omega_2)t}}_{\text{sum-frequency generation}} + \underbrace{E_1 E_2^* e^{-i(\omega_1-\omega_2)t}}_{\text{difference frequency generation}} + c.c\right]$$



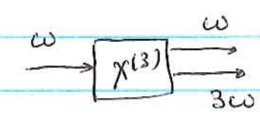
Parametric down conversion

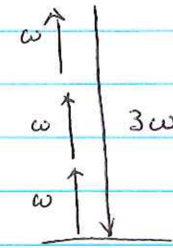


often a cavity is used to enhance the nonlinearity

Third-order nonlinearity

$$P^{(3)} = \chi^{(3)} E^3$$





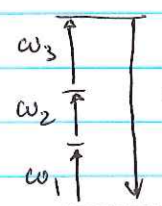
Third harmonic generation

Four-wave mixing

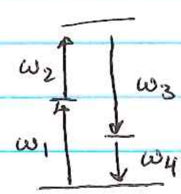
$$\vec{E}(t) = E_1 e^{i\omega_1 t} + E_2 e^{i\omega_2 t} + E_3 e^{i\omega_3 t} + c.c$$

The resulting 3^{rd} order polarization has 22 terms at different frequencies.

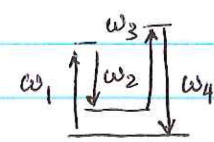
For example



$\omega_4 = \omega_1 + \omega_2 + \omega_3$



$\omega_4 = \omega_1 + \omega_2 - \omega_3$



$\omega_4 = \omega_1 - \omega_2 + \omega_3$

So how do we know which frequencies will be produced?

Propagation equations

$$E_1(z,t) = \mathcal{E}_1 e^{ik_1 z - i\omega_1 t} + c.c \qquad E_2(z,t) = \mathcal{E}_2 e^{ik_2 z - i\omega_2 t} + c.c$$

Newly generated field

$$E_3 = \mathcal{E}_3 e^{ik_3 z - i\omega_3 t}$$

$$ik_3 z$$

$$P_3(z,t) = \underbrace{\chi^{(2)} \mathcal{E}_1 \mathcal{E}_2 e^{i(k_1+k_2)z}}_{\text{slowly varying amplitude}} \cdot e^{-i(\omega_1+\omega_2)t + ik_3 z}$$

slowly varying amplitude $\mathcal{P}_3(z,t)$

Propagation equation for slowly-varying amplitudes

$$\frac{\partial \mathcal{E}_3}{\partial z} + \cancel{\frac{1}{c}\frac{\partial \mathcal{E}_3}{\partial t}} = \frac{ik_3}{2\varepsilon_0}\mathcal{P}$$

$=0$ in a steady-state conditions

$$\frac{d\mathcal{E}_3}{dz} = \frac{ik_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_1\mathcal{E}_2 e^{i(k_1+k_2)z - ik_3 z} = \frac{ik_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_1\mathcal{E}_2 e^{i\Delta k z}$$

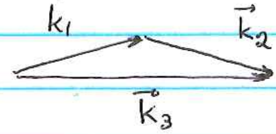
$\Delta k = k_1 + k_2 - k_3$ — momentum mismatch

If $\Delta k = 0$ — phase-matching conditions

$$k_{1z} + k_{2z} - k_{3z} = 0$$

$$\frac{n(\omega_1)\cdot\omega_1}{c}\cos\theta_1 + \frac{n(\omega_2)\cdot\omega_2}{c}\cos\theta_2 = \frac{n(\omega_3)\,\omega_3}{c}\cos\theta_3$$

— momentum conservation



can be achived geometrically, or using tunability of a refractive index.

— energy conservation

$$\omega_1 + \omega_2 = \omega_3$$

If $\Delta k \neq 0$

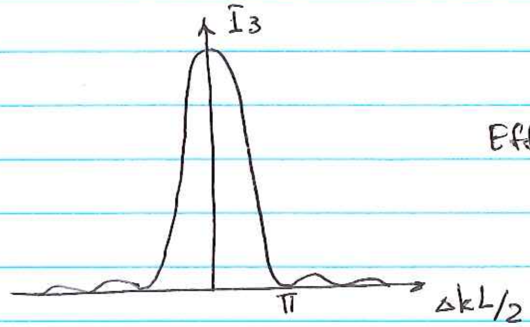
$$\frac{d\mathcal{E}_3}{dz} = \frac{ik_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_1\mathcal{E}_2 e^{i\Delta k z}$$

If $\mathcal{E}_1$ and $\mathcal{E}_2$ do not change much (undepleted pump approximation)

$$\mathcal{E}_3(z) = \frac{ik_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_1\mathcal{E}_2 \int_0^L e^{i\Delta k z}\,dz = \frac{k_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_1\mathcal{E}_2 \frac{e^{-i\Delta k L}-1}{\Delta k}$$

Intensity of the generated field $\propto |\mathcal{E}_3|^2$

$$I_3 \propto I_1 I_2 (\chi^{(2)})^2 L^2 \underbrace{\frac{\sin^2 \Delta k L/2}{(\Delta k L/2)^2}}_{\mathrm{sinc}(\Delta k L/2)}$$



Efficient generation only if $\Delta k L \sim 1$

For phase-matched conditions

$$\frac{d\mathcal{E}_3}{dz} = \frac{ik_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_1\mathcal{E}_2 \quad \Rightarrow \quad \mathcal{E}_3(z) = \frac{ik_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_1\mathcal{E}_2 \cdot z$$

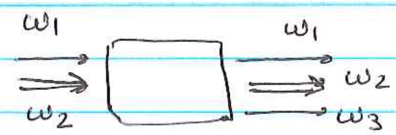
linear growth

This is a good solution as long as the energy transfered from E_1 and E_2 is small compare to these two fields. Then, the undepleted pump approximation breaks, and changes in all three fields must be accounted for.

Also, if there are optical losses α_3 for E_3

$$\frac{d\mathcal{E}_3}{dz} = \underbrace{-\alpha_3\mathcal{E}_3}_{\text{loss}} + \underbrace{\frac{ik_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_1\mathcal{E}_2}_{\text{gain}}$$

Up-conversion (sum-frequency generation with one strong and one weak field)

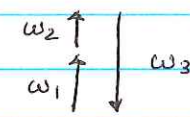


$\omega_3 = \omega_1 + \omega_2$

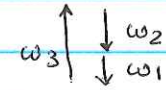
ω_2 – pump
ω_1 – signal
ω_3 – idler

In this case $\mathcal{E}_1$ and $\mathcal{E}_3$ are weak and changing, and $\mathcal{E}_2$ is ~~the~~ strong and constant

$$\frac{d\mathcal{E}_3}{dz} = \frac{ik_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_1\mathcal{E}_2 e^{i\Delta k z}$$



$$\frac{d\mathcal{E}_1}{dz} = \frac{ik_1}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_3\mathcal{E}_2^{*} e^{-i\Delta k z}$$



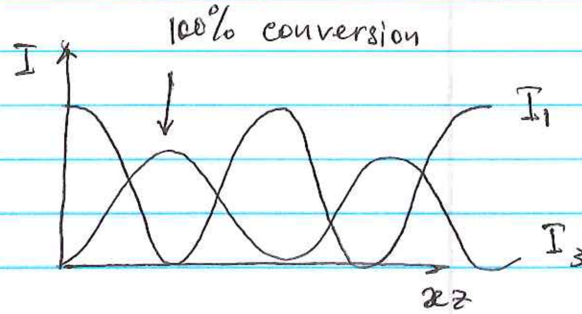
if $\Delta k = 0$ (perfect phase matching)

$$\frac{d^2\mathcal{E}_3}{dz^2} = \frac{ik_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_2\frac{d\mathcal{E}_1}{dz} = -\underbrace{\frac{k_1k_3}{4\varepsilon_0^2}(\chi^{(2)})^2|\mathcal{E}_2|^2}_{\kappa^2}\mathcal{E}_3$$

same for $\mathcal{E}_1$

$$\mathcal{E}_1(z) = \mathcal{E}_1(0)\cos\kappa z$$

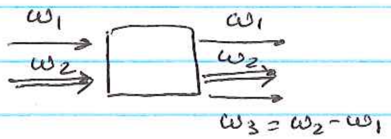
$$\mathcal{E}_3(z) = -\mathcal{E}_1(0)\frac{\kappa}{\frac{ik_1}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_2^{*}}\sin\kappa z \quad \cancel{\Delta 2\mathcal{E}_1(0)\frac{ik_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_2\sin\kappa z}$$

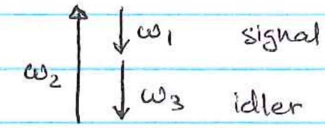


$$\frac{\kappa}{\frac{ik_1}{2\varepsilon_0}\chi^2\mathcal{E}_2^{*}} = \frac{\sqrt{k_1k_3}\,|\mathcal{E}_2|}{i\,k_1\,\mathcal{E}_2^{*}} = \frac{|\mathcal{E}_2|}{i\mathcal{E}_2^{*}}\sqrt{k_3/k_1}$$

pure phase

Frequency - difference generation





$$\frac{d\mathcal{E}_3}{dz} = \frac{ik_3}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_2\mathcal{E}_1^* \cdot \cancel{e^{i\Delta kz}}$$

$$\frac{d\mathcal{E}_1}{dz} = \frac{ik_1}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_2\mathcal{E}_3^* \; \cancel{e^{i\Delta kz}}$$

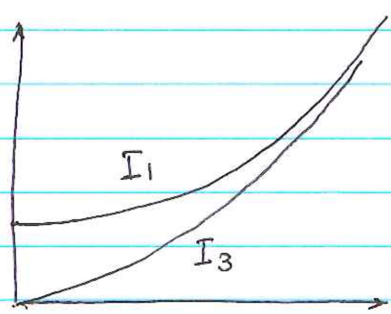
$$\downarrow \quad \frac{d\mathcal{E}_1^*}{dz} = -\frac{ik_1}{2\varepsilon_0}\chi^{(2)}\mathcal{E}_2^*\mathcal{E}_3$$

$$\frac{d^2\mathcal{E}_3}{dz^2} = +\frac{k_3k_1}{4\varepsilon_0^2}(\chi^{(2)})^2|\mathcal{E}_2|^2\mathcal{E}_3 = \varkappa^2\mathcal{E}_3$$

$$\frac{d^2\mathcal{E}_1}{dz^2} = \frac{k_1k_3}{2\varepsilon_0^2}(\chi^{(2)})^2|\mathcal{E}_2|^2\mathcal{E}_1 = \varkappa^2\mathcal{E}_1$$

$$\mathcal{E}_1 = \mathcal{E}_1(0)\cosh\varkappa z$$

$$\mathcal{E}_3 = \mathcal{E}_1^*(0)\left(i\frac{\mathcal{E}_2}{|\mathcal{E}_2|}\sqrt{\frac{k_3}{k_1}}\right)\sinh\varkappa z$$



Both fields grow simultaneously (and exponentially) until the undepleted pump approximation breaks

Parametric amplification