Classical model of an atom

We assume that e-m field does not dramatically affects the atomic structure
→ ~~per~~ weak perturbation

We assume that in the absence of e-m field the dipole moment of an atom is zero, and the induced oscillations of an electron (charge density) produces oscillating dipole
⇒ Classical atom = linear harmonic oscillator
For simplicity – one dimentional ~~osci~~ SHO

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{e}{m}E(t) \qquad E(t) = E_0\cos\omega t$$

ω_0 – "natural" oscillator frequency
γ – damping constant
(Big weakness of the classical model – no way to explain the values of these parameters)

Forced harmonic oscillator

$$E(t) = \frac{1}{2}E_0 e^{-i\omega t} + c.c$$

(since $a_0 \ll \lambda$, the position dependence is not important)

~~So, let's say $x(t) = \frac{1}{2}x_0 X(t) e^{-i\omega t} + c.c.$ where $X(t) =$~~

Solution

$$x(t) = \frac{i}{2}\,\frac{eE_0}{2m\omega}\,\frac{e^{-i\omega t}}{\gamma + i(\omega_0^2 - \omega^2)/2\omega} + c.c$$

For vast majority of systems $\gamma \ll \omega_0$, thus the responce of an atom to e-m field is the strongest for $\omega \approx \omega_0$

assuming $|\omega - \omega_0| \ll \omega, \omega_0$

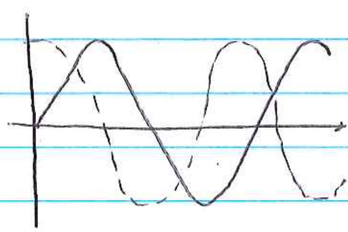
$$\omega_0^2 - \omega^2 \approx (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega(\omega_0 - \omega)$$

$$x(t) = \frac{i}{2} \frac{eE_0}{2m\omega} \frac{e^{-i\omega t}}{\gamma + i(\omega_0 - \omega)} + c.c$$

Exactly on resonance $(\omega = \omega_0)$

$$x(t) = \frac{i}{2} \frac{eE_0}{2m\omega\gamma} e^{-i\omega t} - \frac{i}{2} \frac{eE_0}{2m\omega\gamma} e^{i\omega t} = \frac{eE_0}{2m\omega\gamma} \sin\omega t$$

Notice the phase difference in the oscillator responce



Near, but not exactly on resonance the amplitude of $x^{(+)}(t)$ is a complex number, so the phase delay b/w $x(t)$ and $E(t)$ varies as a function of e-m field detuning $(\omega - \omega_0)$

For convenience

$$x(t) = x_0(t)\, e^{-i\omega t} + c.c$$

where $x_0(t)$ is a slowly-varying amplitude (assuming that $E_0(t)$ is the slow-varying amplitude for e-m field)

$$x_0(t) = \frac{i}{2} \frac{e}{2m\omega} E_0(t) \frac{1}{\gamma + i(\omega_0 - \omega)}$$

↑ at the atom's location

Maxwell's equations in a medium

$$-\nabla^2 \vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = -\mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$$

$$\vec{E}(z,t) = \frac{1}{2}\vec{E}_0(z,t)\, e^{ikz-i\omega t+\varphi} + c.c$$

$$\vec{E}_0 \parallel \vec{e}_x$$

Since the polarization in the material is created by the external e-m field, the oscillations occur at the same frequency (we assume the regime of a linear responce)

$$\vec{P} = \sum_{\substack{\text{unit}\\\text{volume}}} \langle \vec{d}_i \rangle = N \langle \vec{d}(t) \rangle_{\substack{\text{average over}\\\text{all atoms}}}$$

Since we know $\vec{d}(t) = -e\vec{r}$ electron displacement

In a 1D model

$$d = -e x(t)$$

So it is convenient to break $\vec{P}$ into positive and negative frequency parts

$$\vec{P}(z,t) = \vec{P}^+(z,t) + \vec{P}^-(z,t)$$

$$\vec{P}^+ = \frac{1}{2}\vec{e}_x\, \mathcal{P}(z,t)\, e^{ikz-i\omega t+\varphi(z,t)}$$

$$\mathcal{P}(z,t) = N(-e)x_0 = \frac{e^2 N}{2i\, m\omega}\,\frac{1}{\gamma + i(\omega_0-\omega)}\, E_0(z,t)$$

Even if $E_0(z,t)$ is a real function, $\mathcal{P}(z,t)$ in general is not, so we have to be careful with its real and imaginary part

$$\mathcal{P}(z,t) = \varepsilon_0(\chi' + i\chi'')\, E_0(z,t)$$

$\chi = \chi' + i\chi''$ – linear susceptibility of the medium

Using same treatment as before, we can obtain the slow-varying amplitude and phase equations $\left(\left|\frac{\partial P_0}{\partial t}\right| \ll \omega |P_0|, \text{etc}\right)$

$$\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t} = -\frac{k}{2\varepsilon_0} \operatorname{Im}(P) = -\frac{k}{2} \chi'' E_0(z,t)$$

$$E_0\left(\frac{\partial \varphi}{\partial z} + \frac{1}{c}\frac{\partial \varphi}{\partial t}\right) = +\frac{k}{2\varepsilon_0} \operatorname{Re}(P) = +\frac{k}{2}\chi' E_0(z,t)$$

or $$\frac{\partial \varphi}{\partial z} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = +\frac{k}{2}\chi'$$

To better see the physical meaning of χ' & χ'', let's consider special cases.

1. Steady-state e-m field (i.e $E_0(z,t) \neq E_0(t)$)

$$\frac{\partial E_0}{\partial t} = 0$$

$$\frac{\partial E_0}{\partial z} = -\frac{k}{2}\chi'' E_0$$

$$E_0(z) = E_0(0)\, e^{-k\chi''/2 \cdot z} = \underline{E_0(0)\, e^{-\alpha z}}$$

Beer's law

α - absorption coefficient

$$\alpha = \frac{1}{2}k\chi''$$

$$\frac{\partial \varphi}{\partial t} = 0 \qquad \frac{\partial \varphi}{\partial z} = +\frac{k\chi'}{2} \qquad \varphi(z) = \varphi(0) + \frac{k\chi' z}{2}$$

Total phase

$$kz - \omega t + \varphi(z) = \underbrace{k\left(1 + \frac{\chi'}{2}\right)}_{n \text{ - refractive index}} z - \omega t + \varphi_0$$

From the oscillator model

$$\chi(\omega) = \frac{N}{i\varepsilon_0} \frac{e^2}{2m\gamma\omega_0} \frac{\gamma}{\gamma + i(\omega_0 - \omega)}$$

Complex absorption amplitude

$$\alpha(\omega) = \frac{i}{2} k\, \chi(\omega) = \boxed{\frac{kN}{2\varepsilon_0} \frac{e^2}{2m\gamma\omega}} \frac{\gamma}{\gamma + i(\omega_0 - \omega)}$$

α_0 − resonant absorption coefficient

($\alpha(\omega) = \alpha_0$ for $\omega = \omega_0$)

$$\chi(\omega), \alpha(\omega) \propto \frac{\gamma}{\gamma + i(\omega_0 - \omega)} = \frac{\gamma(\gamma - i(\omega_0 - \omega))}{\gamma^2 + (\omega_0 - \omega)^2}$$

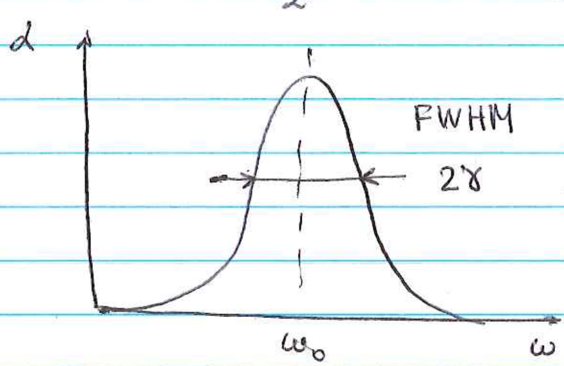
$$\chi''(\omega) = \chi''(\omega_0) \frac{\gamma^2}{\gamma^2 + (\omega_0 - \omega)^2}$$

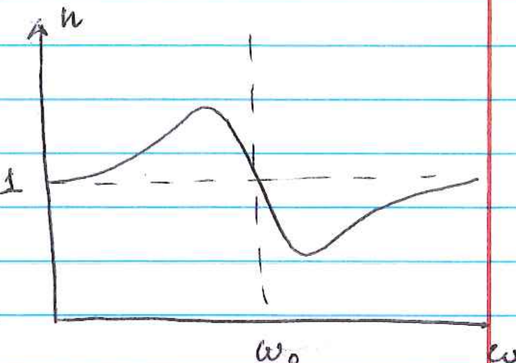
$$\mathrm{Re}(\alpha(\omega)) = \alpha_0 \frac{\gamma^2}{\gamma^2 + (\omega_0 - \omega)^2}$$

absorption

$$\chi'(\omega) = \chi'(\omega_0) \frac{\gamma(\omega_0 - \omega)}{\gamma^2 + (\omega_0 - \omega)^2}$$

$$n(\omega) = 1 + \frac{1}{2}\chi'(\omega) = 1 + \frac{1}{2}\chi'(\omega_0) \frac{\gamma(\omega_0 - \omega)}{\gamma^2 + (\omega_0 - \omega)^2}$$





Radiative damping

Moving electric charge emits e-m radiation
$\rightarrow$ it looses energy $\rightarrow$ oscillations die out.

Oscillating electron emits "self-field" $E_s(\vec{r}, t)$

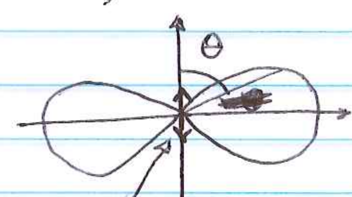
$$\vec{E}_s(r,t) = \frac{e}{4\pi\varepsilon_0 c^2}\left(\frac{\vec{n}\times(\vec{n}\times\dot{\vec{v}})}{r}\right)_{t-r/c} \qquad \vec{n} = \frac{\vec{r}}{r}$$

$$B_s(r,t) = \frac{1}{c}\,\vec{n}\times\vec{E}_s(r,t)$$

Poynting vector $\vec{S}$ ($\cdot$ $\vec{S} = \frac{1}{\mu_0}\vec{E}_s\times\vec{B}_s$)

$$\vec{S} = \frac{1}{\mu_0 c}(\vec{E}_s\cdot\vec{E}_s)\,n = \frac{e^2}{16\pi^2\varepsilon_0^2 c^5 \mu_0}\,\frac{1}{r^2}\,(\vec{n}\times\dot{\vec{v}})^2\,\vec{n} =$$

$$= \frac{e^2|\dot{v}|^2\sin^2\theta}{16\pi^2\varepsilon_0 c^3 r^2}\,\vec{n}$$



oscillating charge

Total emitted ~~energy~~ power

$$\int \vec{S}\,d\vec{a} = \frac{2}{3}\,\frac{e^2}{4\pi\varepsilon_0 c^3}\,|\dot{v}|^2$$

We assume that all emitted energy was drawn from the oscillating electron by a self-induced radiative force

$$\int_t^{t+\Delta t} \vec{F}_{rad}\,\vec{v}\,dt' = -\int_t^{t+\Delta t}\left(\frac{2}{3}\,\frac{e^2}{4\pi\varepsilon_0 c^3}\right)\dot{\vec{v}}\cdot\dot{\vec{v}}\,dt'$$

$$\int_t^{t+\Delta t} \dot{\vec{v}}\,\dot{\vec{v}}\,dt' = \underbrace{\dot{\vec{v}}\,\vec{v}\Big|_t^{t+\Delta t}}_{\text{can neglect for } \Delta t \gg 1/\omega} - \int_t^{t+\Delta t} \vec{v}\cdot\ddot{\vec{v}}\,dt'$$

$$\int_t^{t+\Delta t} \vec{F}_{rad} \vec{v}\, dt' = \int_{t_1}^{t+\Delta t} \frac{2}{3} \frac{e^2}{4\pi\varepsilon_0 c^3} \ddot{\vec{v}} \cdot \vec{v}\, dt'$$

$$\vec{F}_{rad} = \frac{2}{3} \frac{e^2}{4\pi\varepsilon_0 c^3} \ddot{\vec{v}} = -\frac{2}{3} \frac{e^2\omega^2}{4\pi\varepsilon_0 c^3} \vec{v}$$

This is equivalent to the "friction" damping term in the equation of motion for a harmonic oscillator

$$\gamma = \frac{1}{4\pi\varepsilon_0} \frac{1}{3} \frac{e^2\omega^2}{c^3 m}$$