Classical electrodynamics

Light is an electromagnetic wave.

We "know" this from Maxwell equs

no sources
$$\begin{cases} \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & \nabla\times\vec{H} = \dfrac{\partial\vec{D}}{\partial t} \\ \nabla\cdot\vec{B} = 0 & \nabla\cdot\vec{D} = 0 \end{cases}$$

ε - permittivity
μ - permeability

$\vec{E}$ - electric field

$\vec{D}$ - electric displacement

$\vec{B}$ - magnetic induction

$\vec{H}$ - magnetic field

$$\left.\begin{array}{l}\vec{D} = \varepsilon_0\varepsilon\vec{E} \\ \vec{B} = \mu_0\mu H\end{array}\right]\text{ isotropic medium}$$

$$\varepsilon_0\mu_0 = 1/c^2$$

Quantum electrodynamics: $\hat{\vec{E}}, \hat{\vec{D}}, \hat{\vec{B}}, \hat{\vec{H}}$

Each classical field corresponds to an operator

for example,

$$\vec{E} = \langle\psi|\hat{\vec{E}}|\psi\rangle$$

Quantum fields must obey Maxwell equations as well! (since they are linear)

$$\nabla\times\langle\psi|\hat{\vec{E}}|\psi\rangle = -\frac{\partial\langle\psi|\hat{\vec{B}}|\psi\rangle}{\partial t}$$

$$\langle\psi|\nabla\times\hat{\vec{E}} + \frac{\partial\hat{\vec{B}}}{\partial t}|\psi\rangle = 0 \qquad \text{for } \forall\psi$$

$$\nabla\times\hat{\vec{E}} + \frac{\partial\hat{\vec{B}}}{\partial t} = 0$$ (and so on for other equations)

Similarly $\hat{\vec{D}} = \varepsilon_0\varepsilon\hat{\vec{E}}$, $\hat{\vec{B}} = \mu_0\mu\hat{\vec{H}}$

Classical e-m field energy

$$H = \frac{1}{2}\int dV\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)$$

Corresponding QED Hamiltonian

$$\hat{H} = \frac{1}{2}\int dV\left(\hat{\vec{E}}\cdot\hat{\vec{D}} + \hat{\vec{B}}\cdot\hat{\vec{H}}\right) = \frac{1}{2}\int dV\left(\varepsilon_0\varepsilon\,\hat{\vec{E}}^2(\vec{r},t) + \frac{1}{\mu_0\mu}\hat{\vec{B}}^2(\vec{r},t)\right)$$

$\hat{\vec{E}}$ and $\hat{\vec{B}}$ are not independent, but connected through Maxwell eqns.

Vector potential $\hat{\vec{A}}$

$\hat{\vec{E}} = -\partial\hat{\vec{A}}/\partial t \qquad \hat{\vec{B}} = \nabla\times\vec{A}$

Coulumb gauge: $\nabla(\varepsilon\vec{A}) = 0$ [and so $\nabla\vec{D} = 0$].

wave equation for $\hat{\vec{A}}$

$$\frac{1}{\varepsilon}\nabla\times\frac{1}{\mu}\nabla\times\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0$$

isotropic medium $\varepsilon \neq \varepsilon(\vec{r}),\ \mu \neq \mu(\vec{r})$

$$\nabla^2\hat{\vec{A}} + \frac{\mu\varepsilon}{c^2}\frac{\partial^2\hat{\vec{A}}}{\partial t^2} = 0$$

Classical problems:

Typical way to proceed: consider boundary conditions for the problem, find a good mode basis that satisfies these boundary conditions $\vec{A}_k(\vec{r},t)$, and then figure out what combinations of these modes provides the right solution

$$\vec{A}(\vec{r},t) = \sum_k c_k\,\vec{A}_k(\vec{r},t)$$

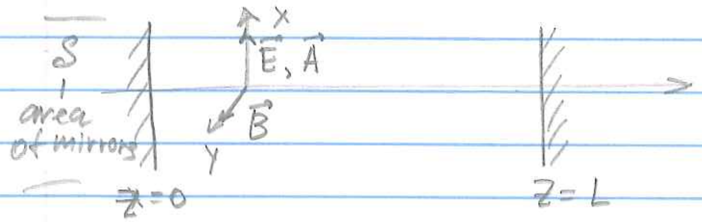
QED solution:

$$\hat{\vec{A}}(\vec{r},t) = \sum_k\left[\vec{A}_k(\vec{r},t)\,\hat{a}_k + \vec{A}_k^*(\vec{r},t)\,\hat{a}_k^+\right]$$

where $\hat{a}_k, \hat{a}_k^+$ are lowering and raising operators for SHO

(usually called annihilation and creation operators in quantum optics)

Simplest case: plane EM wave b/w two perfect conductors



$\mathcal{E}=1 \quad \mu=1$

$\left[\begin{array}{l}\vec{E}(z=0)=\vec{E}(z=L)=0 \\ \text{boundary conditions}\end{array}\right.$

$$\nabla^2\vec{A}+\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t}=0 \quad \text{or} \quad \nabla^2\vec{E}+\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t}=0$$

Spatial modes that automatically satisfy the boundary conditions: standing waves

$$\vec{A},\vec{E} \propto \sin\left[\frac{\pi n z}{L}\right] \qquad n=1,2,\ldots$$

Single - mode approximation
(one particular standing wave is excited)
Let's pick $\vec{E}=(E_x,0,0)$, $\vec{B}=(0,B_y,0)$, $\vec{A}=(A_x,0,0)$

$$E_x(\vec{r},t)=E_x(z,t)=\mathcal{E}(t)\cdot\sin(kz) \qquad k=\frac{\pi n}{L}=\frac{\omega}{c}$$

At that point we are going to "guess" the normalization

$$E_x(z,t)=\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\,q(t)\,\sin kz \qquad V\text{ - volume of the cavity}$$

$$\nabla\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \qquad \frac{\partial B_y}{\partial z}=\frac{1}{c^2}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\,\dot{q}(t)\,\sin kz$$

$$B_y=-\frac{1}{\underset{\omega}{(ck)}}\sqrt{\frac{2\omega^2}{V\varepsilon_0 c^2}}\,\dot{q}(t)\cos kz=-\sqrt{\frac{2}{V\varepsilon_0 c^2}}\,\dot{q}(t)\cos kz$$

When moving to quantum case
$q\to\hat{q}$, $\dot{q}\to\hat{p}$ (canonical position and momentum)

Hamiltonian

$$H = \frac{1}{2}\int dV \left[\varepsilon_0 E_x^2 + \frac{1}{\mu_0} B_y^2 \right] =$$

$$= \frac{1}{\cancel{2}} \varepsilon_0 \left(\frac{\cancel{2}\omega^2}{V\varepsilon_0} \right) q^2(t) \int dV \cdot \sin^2 kz + \frac{1}{\cancel{2}\mu_0} \frac{\cancel{2}}{V\varepsilon_0} (\dot{q}(t))^2 \int dV \cos^2 kz$$

$$\int_V \sin^2 kz \; dz\,dx\,dy = S \int_0^L \sin^2 kz \, dz = S \cdot \frac{L}{2} = \frac{V}{2}$$

$$H = \frac{\omega^2}{\cancel{V}} q^2(t) \cdot \frac{\cancel{V}}{2} + \frac{1}{\cancel{V}} (\dot{q})^2 \frac{\cancel{V}}{2} = \frac{1}{2}\left((\dot{q})^2 + \omega^2 q^2 \right)$$

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \omega^2 \hat{q}^2 \right) \qquad \text{SHO !}$$

$$[\hat{q}, \hat{p}] = i\hbar$$

Annihilation and creation operators

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}} (\omega\hat{q} + i\hat{p}) \qquad \hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega}} (\omega\hat{q} - i\hat{p})$$

$$\left[\begin{array}{l} \hat{E}_x = \sqrt{\dfrac{\hbar\omega}{\varepsilon_0 V}} (\hat{a} + \hat{a}^+) \sin kz \\ \hat{B}_y = \dfrac{1}{c}\sqrt{\dfrac{\hbar\omega}{\varepsilon_0 V}} (\hat{a} - \hat{a}^+) \cos kz \end{array} \right.$$

$$[a, a^+] = 1 \longrightarrow aa^+ - a^+a = 1$$

$$\hat{H} = \hbar\omega \left(\hat{a}^+\hat{a} + \frac{1}{2} \right)$$

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}] = i\omega [\hat{a}^+\hat{a}, \hat{a}] = i\omega \left(\hat{a}^+\hat{a}\hat{a} - \hat{a}\overbrace{\hat{a}\hat{a}^+}^{1+\hat{a}^+\hat{a}}\hat{a} \right)$$

$$= -i\omega\hat{a}$$

$$\hat{a}(t) = e^{-i\omega t}\hat{a}$$

Plane e-m linearly polarized wave inside the cavity

$$\begin{cases} \hat{E}_x = \sqrt{\dfrac{\hbar\omega}{\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t} + \hat{a}^+ e^{i\omega t}\right) \sin kz \\ \hat{B}_y = \dfrac{1}{c}\sqrt{\dfrac{\hbar\omega}{\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t} - \hat{a}^+ e^{i\omega t}\right) \cos kz \end{cases}$$

Fock states (or number states)

$\hat{n} = \hat{a}^+\hat{a}$ $\qquad \hat{n}|n\rangle = n|n\rangle$

$|n\rangle$ - eigenstates of the number operator, describing the state with known number of photons

$|0\rangle$ - vacuum state (no photons)
$|1\rangle$ - single-photon state

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)|n\rangle = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)|n\rangle = E_n|n\rangle$$

$$E_n = n\hbar\omega + \frac{\hbar\omega}{2}$$

Why $\hat{a}$ is annihilation operator?

$$\hat{n}(\hat{a}|n\rangle) = \overbrace{\hat{a}^+\hat{a}}^{\hat{a}\hat{a}^+ - 1}(\hat{a}|n\rangle) = (\hat{a}\underbrace{\hat{a}^+\hat{a}}_{\hat{n}} - \hat{a})|n\rangle =$$

$$= \hat{a}\underbrace{(\hat{n}|n\rangle)}_{= n|n\rangle} - \hat{a}|n\rangle = (n-1)\hat{a}|n\rangle \qquad \left[\text{one photon is gone after } \hat{a}\right]$$

$\hat{a}|n\rangle$ is an eigenstate of $\hat{n}$ with eigen value of $(n-1)$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

Coherent states

$$E_x = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\hat{a} e^{ikz-i\omega t} + \hat{a}^+ e^{-ikz+i\omega t}\right)$$

The number states we've discussed are highly non-classical, i.e. they do not have classical analogues

Measured mean amplitude of e-m field

$$\langle E_x\rangle = \langle n|E_x|n\rangle = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\Big(\underbrace{\langle n|\hat{a}|n\rangle}_{=0} e^{ikz-i\omega t} - \underbrace{\langle n|\hat{a}^+|n\rangle}_{=0} e^{-ikz+i\omega t}\Big)$$

$= 0$ no well-defined electric field

Fluctuations of electro-magnetic field

$$\Delta E_x = \sqrt{\langle E_x^2\rangle - |\langle E_x\rangle|^2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\sqrt{\langle n|\,|\hat{a}e^{ikz-i\omega t} - \hat{a}^+ e^{-ikz+i\omega t}|^2|n\rangle}$$

$$= \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\sqrt{\langle n|\,\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\,|n\rangle} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\sqrt{2n+1}$$

Note that even for $|0\rangle$ vacuum state

$$\Delta E_x = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} > 0$$

Vacuum fluctuations (now vacuum is not nothingness, it is alive and wiggles)

So what state would be the closest analogue of the classical e-m wave?

Vacuum fluctuations

For a single mode $E_n = \hbar\omega(n+\frac{1}{2})$
For a vacuum state $n=0$
Zero point energy
$$E_{ZPE} = \sum_{mode} \frac{1}{2}\hbar\omega \longrightarrow \infty$$

(Too) Simple solution → renormalization (just shift the level from which we count energy)
The effect of fluctuations is directly observable

a) Spontaneous emission: electron in excited states interact with vacuum fluctuation and, as a result, change their energy level, emitting thermal radiation

b) Since there is always uncertainty in measurable e-m field amplitude, all optical measurements are fundamentally limited in precision

c) Lamb shift
Experimentally $2S_{1/2}$ and $2P_{1/2}$ states in H atom are split by ~ 1 GHz
In a semiclassical approximation, they must be degenerate.

Vacuum fluctuations make an electron to randomly fluctuate from its equilibrium position, changing its energy in the Coulomb potential

$$\Delta E = \frac{1}{6} \langle (\Delta r)^2 \rangle \cdot 4\pi e^2 \underbrace{|\psi_{nml}(r=0)|^2}_{=0 \text{ for all states except for } l=0 \text{ (S state)}}$$

d) Casimir force

$$E_{ZPE} = \sum_{\text{modes}} \frac{1}{2} \hbar \omega_i$$

Two perfectly conducting parallel plates

if $d \sim \lambda$, only the wavevector $k_z = \frac{\pi n}{d}$ are possible

$$\omega_n = c|k|_n = c\sqrt{k_x^2 + k_y^2 + (\pi n/d)^2}$$

$$E_{ZPE}^{(in)} = \int dk_x dk_y \sum_n \frac{1}{2} \hbar c \sqrt{k_x^2 + k_y^2 + \left(\frac{\pi n}{d}\right)^2}$$

outside $\rightarrow$ no restrictions

$$E_{ZPE}^{(out)} = \int dk_x dk_y dk_z \left(\frac{1}{2} \hbar c \sqrt{k_x^2 + k_y^2 + k_z^2}\right)$$

$$U = E_{ZPE}^{(in)} - E_{ZPE}^{(out)} = \{\text{after tedious calculations}\}$$

$$= \frac{\pi^2 \hbar c}{720 d^3} L^2$$

Casimir force (per unit area)

$$F = \frac{1}{L^2} \frac{dU}{dd} = - \frac{\pi^2 \hbar c}{240 d^4}$$