Second quantisation

Harmonic oscillator

Classical case

$x(t) = A_1 \cos\omega t + A_2 \sin\omega t$

$p(t) = m\dot{x}(t)$

Energy $= \frac{1}{2} m \frac{p^2}{m} + \frac{1}{2} m\omega^2 x^2$

Quantum case

$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$

$\hat{H}\psi_n = E_n\psi_n$

$E_n = \hbar\omega(n+\frac{1}{2})$

Ladder operators

$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega\hat{x})$

$\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + \frac{1}{2})$

$\hat{a}^+\hat{a}|\psi_n\rangle = n|\psi_n\rangle$

$\hat{a}|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle$

$\hat{a}^+|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle$

Electromagnetic field

$E(t) = E_1\cos\omega t + E_2 \sin\omega t \quad (\lambda \gg a)$

$\frac{1}{c}\nabla\times\vec{B} = \frac{\partial E}{\partial t} \Rightarrow \frac{1}{c}\vec{k}\times\vec{B} = \frac{\partial \vec{E}}{\partial t}; \omega B = \frac{\partial E}{\partial t}$

Energy density: $u = \frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$

similar to SHO

$\hat{H}_{em} = \frac{\varepsilon_0}{2}\hat{E}^2 + \frac{1}{2\mu_0}\hat{B}^2$

$\hat{E} \leftrightarrow \hat{x}$ and $\hat{B} \leftrightarrow \hat{p}$

$\hat{E} = \sqrt{\frac{2\omega^2}{V\varepsilon_0}}\,\hat{x}\ ;\ \hat{B} = \frac{1}{c}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\,\hat{p}$

$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{x}^2)$

$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{x} \pm i\hat{p})$

$\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + \frac{1}{2})$

$\hat{n} = \hat{a}^+\hat{a} \longrightarrow$ number of photons

$|n\rangle$ — a state of e-m field with n photons in it

$$\hat{E} = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left(\hat{a}e^{-i\omega t} + \hat{a}^+e^{+i\omega t}\right) \quad [\times \text{ spatial part}]$$

$$\hat{B} = \frac{1}{c}\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left(\hat{a}e^{-i\omega t} - \hat{a}^+e^{i\omega t}\right) \quad [\times \text{ spatial part}]$$

So now we can interprete the quantized e-m field as a collection of excitations of the quantum SHO, where the photon number stands for the excitation level.

It also means that even in the ground state $(n=0)$ the e-m field will have some energy $(\frac{1}{2}\hbar\omega)$ — quantum vacuum.

Vacuum state $|0\rangle$ $\quad \hat{H}|0\rangle = \hbar\omega(\hat{a}^+\hat{a}|0\rangle + \frac{1}{2}|0\rangle) = \frac{1}{2}\hbar\omega|0\rangle$

Operators $\hat{a}$ and $\hat{a}^+$ will operate the same way as before on the number states

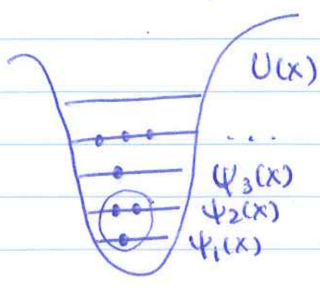
$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$ one photon is destroyed
<u>annihilation</u> <u>operator</u>

$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$ one photon is added
creation operator

Notice that in this formalism we do not discuss the wave function of individual photons, but rather describe our system only by number of photons of a given frequency.

The process of moving from regular wave-functions for individual quantum states into an occupation number representation

First quantization



$$\psi(x_1,x_2,x_3) = \psi_1(x_1)\psi_2(x_2)\psi_2(x_3) +$$
+ all permutations (bosons)

Second quantization

~~Alt~~ Occupation number basis

$|1,2,0,\ldots\rangle$

in general

$|n_1, n_2, n_3 \ldots\rangle$

Number operators

$\hat{n}_i\,|n_1, n_2 \ldots n_i \ldots\rangle = n_i\,|n_1, n_2 \ldots n_i \ldots\rangle$

Main operators: annihilation and creation operators

~~Bosons~~

$$\hat{a}_i\,|n_1, n_2, n_3 \ldots n_i \ldots\rangle = \sqrt{n_i}\,|n_1, n_2 \ldots n_i - 1\rangle$$

$$\hat{a}_i^+\,|n_1, n_2 \ldots n_i \ldots\rangle = \sqrt{n_i+1}\,|n_1, n_2 \ldots n_i + 1 \ldots\rangle$$

Fermions are trickier

$\hat{a}_i |n_i\rangle = \sqrt{n_i}\, |n_i - 1\rangle$ $\qquad \hat{a}_i |0\rangle = 0$ (not physical)

$\qquad\qquad\qquad\qquad\qquad \hat{a}_i |1\rangle = |0\rangle$

$\hat{a}_i^+ |n_i\rangle - ?$ $\qquad \hat{a}_i^+ |0\rangle = 1\, |1\rangle$

$\qquad\qquad\qquad \hat{a}_i^+ |1\rangle = 0$ (not physical)

We have to modify the action of $\hat{a}$ and $\hat{a}^+$ for fermions

$$\hat{a}_i^+ |n_1, n_2 \ldots n_i \ldots\rangle = (-1)^{\sum_{j<i} n_j} (1 - n_i) |n_1, n_2 \ldots 1_i \ldots\rangle$$

$n_i = 0 \rightarrow$ a particle is created, $n_i = 1$

$n_i = 0 \rightarrow$ the resulting expression is zero (not possible)

$$\hat{a}_i |n_1, n_2 \ldots n_i \ldots\rangle = (-1)^{\sum_{j=i} n_j}\, n_i\, |n_1, n_2 \ldots 0_i \ldots\rangle$$

$\sum_{j<i} n_j$ — total number of occupied states with number $j < i$. We need that to keep track of # of permutations for the many-particle wave function and ensure its sign is changed when any two particles are flipped

For example: $\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{2}\left(\psi_1(\vec{r}_a)\psi_2(\vec{r}_b) - \psi_1(\vec{r}_b)\psi_2(\vec{r}_a)\right)$

$|1, 1\rangle \ldots$ or $-|1, 1\rangle$

$\psi_1(\vec{r}_a)\,\psi_2(\vec{r}_b)$ $\qquad\qquad \psi_1(\vec{r}_b)\,\psi_2(\vec{r}_a)$

$\underset{1\ \ 2}{\square\,\square} \xrightarrow{\hat{a}_1^+} \underset{1\ \ 2}{\boxed{a}\,\square} \xrightarrow{\hat{a}_2^+} \underset{1\ \ 2}{\boxed{a}\,\boxed{b}}$ $\qquad \underset{1\ \ 2}{\square\,\square} \xrightarrow{\hat{a}_2^+} \underset{1\ \ 2}{\square\,\boxed{a}} \xrightarrow{\hat{a}_1^+} \underset{1\ \ 2}{\boxed{b}\,\boxed{a}}$

$|0\,0\rangle \rightarrow |1, 0\rangle \rightarrow -|1, 1\rangle$ $\qquad |0\,0\rangle \rightarrow |0\,1\rangle \rightarrow |1\,1\rangle$

$$|n_1, n_2 \ldots n_i\rangle = (\hat{a}_1^+)^{n_1} (\hat{a}_2^+)^{n_2} \ldots (\hat{a}_i^+)^{n_i} |0 \ldots 0\rangle$$

Because of the ~~differences~~ differences in the particle statistics, the creation and annihilation operators ~~have~~ for fermions follow different rules — they anti-commute

$$\{\hat{a}_i^\dagger, \hat{a}_i\} = \hat{a}_i^\dagger \hat{a}_i + \hat{a}\hat{a}_i^\dagger = 1$$

However, we still can use the same definition for the number operator

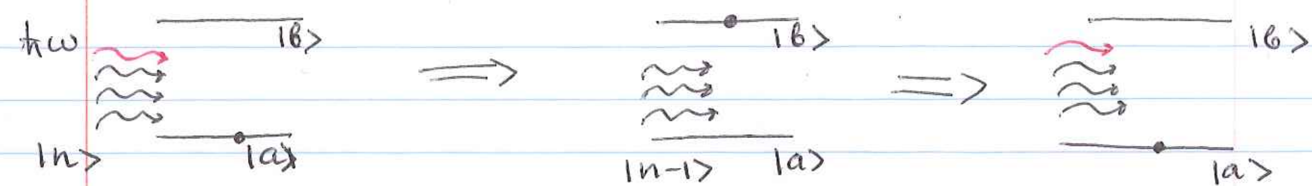
$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$$

$$\hat{n}_i |\ldots n_i \ldots\rangle = \hat{a}_i^\dagger \hat{a}_i |\ldots n_i \ldots\rangle = \hat{a}_i^\dagger \left[(-1)^{\sum_{j<i} n_j} n_i |\ldots 0_i \ldots\rangle \right] =$$

$$= \left((-1)^{\sum_{j<i} n_j} \right)^2 n_i (1-0) \cancel{a_i} |\ldots n_i \ldots\rangle = n_i |\ldots n_i \ldots\rangle$$

Common applications of bosonic second quantization operators

<u>Photon</u> absorption and emission



Absorption: atomic state changes from $|a\rangle \to |b\rangle$
projection operator $|b\rangle\langle a|$
photonic state changes from $|n\rangle \to |n-1\rangle$
operator $\hat{a}_\omega$

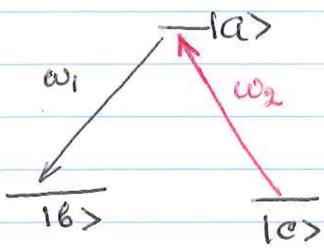
Emission: atomic state operator $|a\rangle\langle b|$
(stimulated) photonic state operator $\hat{a}^+_\omega$

Interaction Hamiltonian (fully quantized)

$$\hat{H} = ig\,\hat{a}\,|b\rangle\langle a| + ig\,\hat{a}^+\,|a\rangle\langle b|$$

g - coupling content, describes the interaction strength

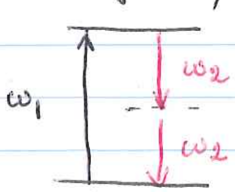
Such notation also helps to understand what processes are described



Two-photon process: one photon of ω_1 is emitted, one photon of ω_2 is absorbed atom is transferred from $|c\rangle$ to $|b\rangle$

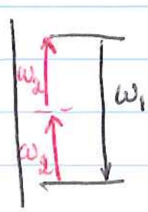
$$\hat{H} = ig\ \hat{a}^+_{\omega_1}\hat{a}_{\omega_2}\ |b\rangle\langle c|$$

Frequency conversion



Parametric down conversion

$$\hat{H}_{PDC} = \alpha\ \hat{a}_{\omega_1}(\hat{a}^+_{\omega_2})^2$$



Second harmonic generation

$$\hat{H}_{SHG} = \alpha\ \hat{a}^+_{\omega_1}\ \hat{a}^2_{\omega_2}$$