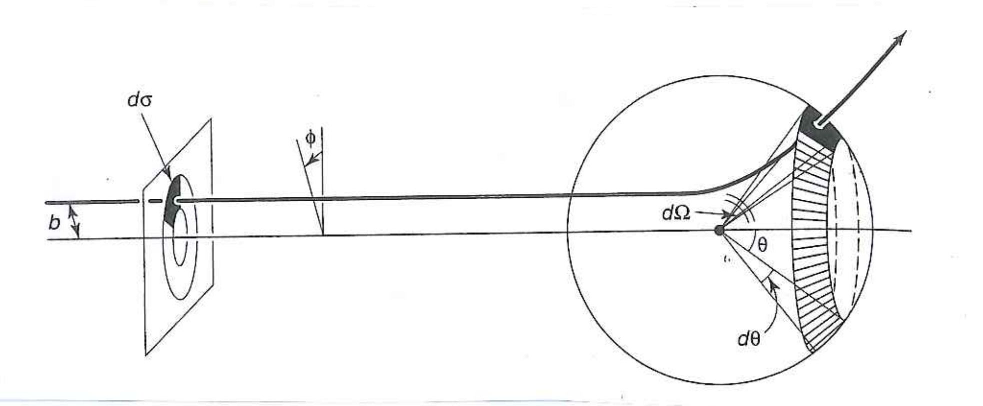
Scattering

Classical scattering

Throw particles at your target, see where they fly

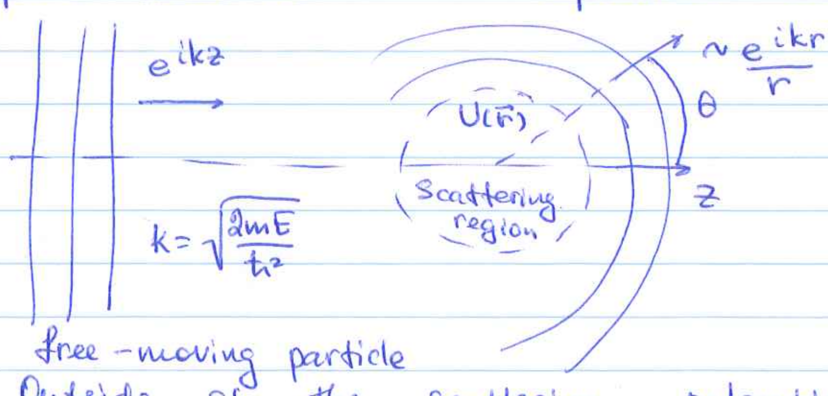


Differential cross-section $\frac{d\sigma}{d\Omega}$

All particles incident in the area $d\sigma$ will scatter into a solid angle $d\Omega$

Quantum scattering (any wave scattering)

plane wave in $\longrightarrow$ spherical wave is scattered out



free-moving particle

Outside of the scattering potential

$$\psi(\vec{r}, \theta) \approx A \left\{ e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \right\}$$

scattering amplitude

Probability of a particle to fall within incident area $d\sigma$ within time dt

$$dP_{inc} = |\psi_{inc}|^2 dV = |A|^2 (v dt) d\sigma$$

Probability to be scattered into solid angle $d\Omega$

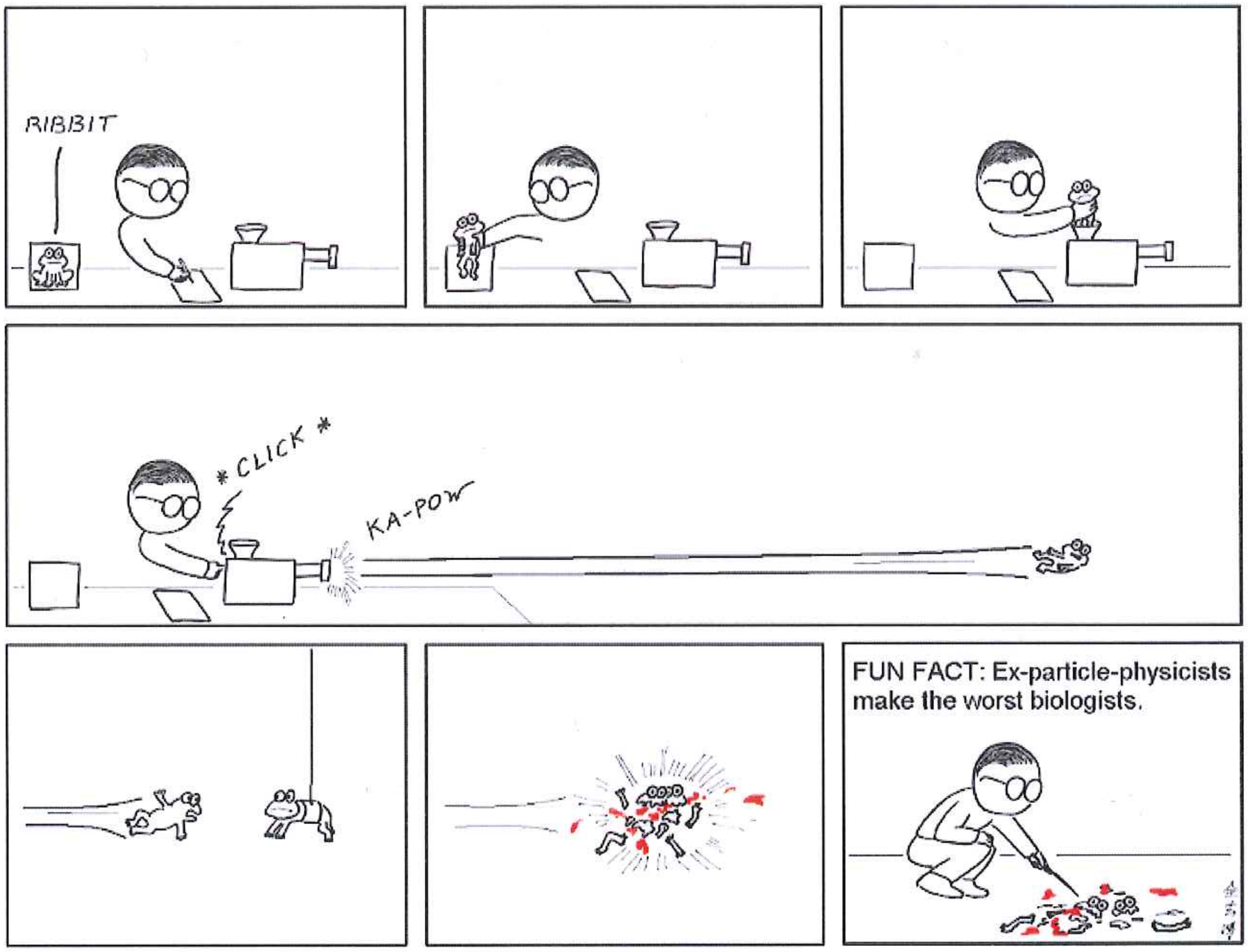
$$dP_{sc} = |\psi_{sc}|^2 dV = |A|^2 |f(\theta)|^2 \frac{1}{\cancel{r^2}} \cdot (vdt) \cancel{r^2} d\Omega =$$

$$= |A|^2 (vdt) |f(\theta)|^2 d\Omega$$

To maintaid $dP_{inc} = dP_{sc}$ $\qquad d\sigma = |f(\theta)|^2 d\Omega$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

~~If one~~ Normally, it is very rare ~~hard to~~ when one needs to find scattering amplitude for known potential, usually one has to solve an inverse problem, and find the scattering potential from measured scattering characteristics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KA-POW
FUN FACT: Ex-particle-physicists make the worst biologists.

Quick recap of Bessel functions
(sine / cosine ugly cousins)

1D oscillations
(free particle)

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

$\sin kx \leftarrow = 0 \ @x=0 \rightarrow j_\ell(r)$ sph Bessel function

$\cos kx \leftarrow \neq 0 \ @x=0 \rightarrow n_\ell(r)$ sph Neumann function

$\Downarrow$

free particle moving in a particular direction

$$e^{\pm ikx} = \cos kx \pm i \sin kx$$

3D oscillations
(free particle)

$$\nabla^2_{r,\theta,\varphi}\psi + k^2\psi = 0$$

$$\psi(r,\theta,\varphi) = R_\ell(r)\, Y_\ell^m(\theta,\varphi)$$

$h_\ell^{(1)} = j_\ell(x) + i n_\ell(x)$ sph. Henkel funct

$h_\ell^{(2)} = j_\ell(x) - i n_\ell(x)$

for $x \gg 1$ ~~for $x \ll 1$~~

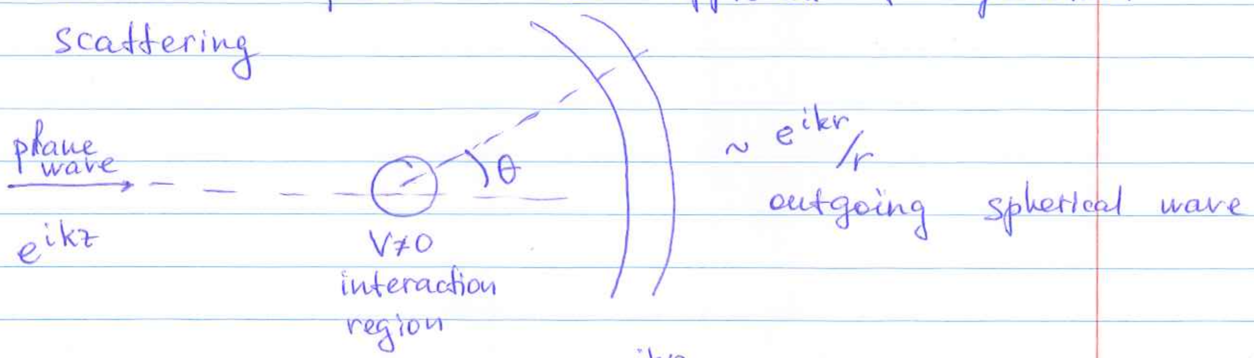
$j_\ell(x) \approx$

$n_\ell(x) \approx$

$$h_\ell^{(1)}(x) \approx \frac{(-i)^{\ell+1}}{x} e^{ix} \qquad h_\ell^{(2)}(x) \approx \frac{i^{\ell+1}}{x} e^{-ix}$$

How to find the scattering amplitude: partial wave analysis

Reminder: ~~problem~~ our approach to quantum scattering



$$\psi(\vec{r}) = A\left(e^{ikz} + \underline{\underline{f(\theta)}}\,\frac{e^{ikr}}{r}\right)$$

Let's find the appropriate form of the solution far from the scattering potential ($V(\vec{r})=0$ there), radiation zone $kr \gg 1$

$V(\vec{r})=0 \longrightarrow$ the particle is free

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,\varphi) = E\psi(r,\theta,\varphi) \Rightarrow \nabla^2\psi(r,\theta,\varphi) + k^2\psi = 0$$

$$k = \sqrt{2mE/\hbar^2}$$

$$\psi(r,\theta,\varphi) = R(r)\,Y_\ell^m(\theta,\varphi) = \frac{u(r)}{r}\,Y_\ell^m(\theta,\varphi)$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}u = Eu$$

$$\frac{d^2u}{dr^2} - \frac{\ell(\ell+1)}{r^2}u = -k^2u \qquad k=\sqrt{\frac{2mE}{\hbar^2}}$$

we expect the solution to be an outgoing spherical wave:

if $\ell(\ell+1) \ll k^2r^2 \Rightarrow \frac{\ell(\ell+1)}{r^2}u \ll k^2u$

$$\frac{d^2u}{dr^2} + k^2u = 0 \longrightarrow \quad u \sim e^{ikr} \qquad R(r) \sim \frac{e^{ikr}}{r}$$

this is the assimptotics of $h_\ell^{(1)}(kr)$!

$$R(r) \sim h_\ell^{(1)}(kr)$$

Since in general, all values of the angular momenta are possible, the general solution must include all possible terms

$$\psi(r,\theta,\varphi) = A\left\{ e^{ikz} + \sum_{\ell,m} c_{\ell m}\, h_\ell^{(1)}(kr)\, Y_\ell^m(\theta,\varphi) \right\}$$

In general, if a ~~po~~ scattering potential has angular dependence, the angular momentum will not be conserved. However, if the potential is spherically symmetric $V(\vec{r}) = V(r)$, and the incoming wave is isotropic $e^{ikz} = e^{ikr\cos\theta}$ (no φ dependence), ~~then~~ and thus ~~it~~ has zero m; thus, we expect that the scattered wave function will also have no φ dependence $(m=0)$

$$Y_\ell^{m=0}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

Traditionally, the wave function is written in the following form

$$\psi(r,\theta,\cancel{\varphi}) = A\left\{ e^{ikz} + k\sum_{\ell=0}^{\infty} i^{\ell+1}(2\ell+1)\, \underline{\underline{a_\ell}}\, h_\ell^{(1)}(kr)\, P_\ell(\cos\theta) \right\}$$

a_ℓ — partial wave amplitude

$$a_\ell = c_{\ell m=0} \Big/ \left(i^{\ell+1}\sqrt{(2\ell+1)\cdot 4\pi}\; k\right)$$

If we know a_l, we can figure out $f(\theta)$

Since $h_l^{(1)}(kr) \to \dfrac{(-i)^{l+1} e^{ikr}}{kr}$

$$k\cdot i^{l+1}(2l+1)a_l h_l^{(1)}(kr) \to \cancel{k}\,\cancel{i^{l+1}}(2l+1)a_l \cancel{h_l^{(1)}(kr)} \cdot \frac{\cancel{(-i)^{l+1}}e^{ikr}}{\cancel{k}r}$$

$$\psi(r,\theta) \xrightarrow{kr\gg 1} A\left\{ e^{ikz} + \left(\sum_{l=0}^{\infty}(2l+1)a_l P_l(\cos\theta)\right)\frac{e^{ikr}}{r}\right\}$$

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l P_l(\cos\theta)$$

$$\frac{d\sigma}{d\Omega} \stackrel{?}{=} |f(\theta)|^2 = \sum_{l,l'=0}^{\infty}(2l+1)(2l'+1)a_l a_{l'} P_l(\cos\theta)P_{l'}(\cos\theta)$$

$$\sigma = \int \frac{d\sigma}{d\Omega}\cdot d\Omega = \sum_{l,l'=0}^{\infty}(2l+1)(2l'+1)a_l a_{l'}^{*} \underbrace{\int P_l(\cos\theta)P_{l'}(\cos\theta)\,d\Omega}_{\frac{4\pi}{2l+1}\delta_{ll'}}$$

$$\sigma = \sum_{l=0}^{\infty} 4\pi(2l+1)|a_l|^2$$

Ok, we have figured out the proper form of the wave function outside of the scattering region

$$\psi(r,\theta) = A\left\{ e^{ikz} + k\sum_{l=0}^{\infty} i^{l+1}(2l+1)a_l h_l^{(1)}(kr) P_l(\cos\theta)\right\}$$

$$e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta)$$

$$\psi(r,\theta) = A \sum_{l=0} i^l (2l+1)\left[j_l(kr) + ik a_l h_l^{(1)}(kr)\right] P_l(\cos\theta)\Big]$$

How to find a_l? Need to find $\psi(r,\theta)$ at the ~~ra~~ scattering region in a form of similar functional expansion, and equate coeffecients.

Hard sphere scattering

$$V(r) = \begin{cases} 0, & r > a \\ \infty, & r < a \end{cases}$$

Our solution is exact everywhere $r > a$

Boundary condition $\psi(r,\theta)\big|_{r=a} = 0$

$$\sum_{l=0}^{\infty} \underbrace{(\ldots\ldots)_l}_{\text{must be } = 0 \text{ for every } l \text{ at } r=a} P_l(\cos\theta) = 0 \qquad \text{for all } \theta$$

$$j_l(ka) + ik\, a_l\, h_l^{(1)}(ka) = 0$$

$$a_l = -\frac{j_l(ka)}{ik\, h_l^{(1)}(ka)}$$

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \left| \frac{j_l(ka)}{h_l^{(1)}(ka)} \right|^2$$

Small particle limit $ka \ll 1$ $\quad \frac{2\pi}{\lambda} a \ll 1$

$$\frac{j_l(ka)}{h_l^{(1)}(ka)} \approx -i \frac{j_l(ka)}{n_l(ka)} = \frac{i}{2l+1} \left[\frac{2l \cdot l!}{(2l)!}\right]^2 (ka)^{2l+1}$$

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} \frac{1}{2l+1} \left[\frac{2^l l!}{(2l)!}\right]^4 (ka)^{4l+2}$$

First-order $\to l=0$ $\qquad \sigma^{(1)} \approx \frac{4\pi}{k^2} \cdot (ka)^2 = 4\pi a^2 \to 4\times$ sphere geometr. cross-section

Second-order $\to l=1$

$$\sigma^{(2)} = \frac{4\pi}{3k^2} (ka)^6 = \frac{4\pi}{3} k^4 a^6 = \frac{2}{3} (2\pi)^5 \frac{a^6}{\lambda^4}$$

same scaling as Rayleigh scattering in optics