Particle distribution function

Last time we calculated the number of possible ways to spread N particles b/w different energy states E_n (each with degeneracy d_n) such than N_n particles are in n-th state.

Distinguishable particles: $W(N_1, N_2 .. N_n ..) = N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!}$

Fermions: $W(N_1, N_2 ..) = \prod_{n=1}^{\infty} \frac{d_n!}{N_n!(d_n - N_n)!}$

Bosons: $W(N_1, N_2 ..) = \prod_{n=1}^{\infty} \frac{(d_n + N_n - 1)!}{N_n!(d_n - 1)!}$

Suppose that we now need to determine the most probable configuration $(N_1, N_2 ...)$ if we know the total energy of the system. Essentially, we need to determine for what combination of $(N_1, N_2 ...)$ $W(N_1, N_2 ..)$ is maximum, with constraints $\sum_{n=1}^{\infty} N_n = N$ and $\sum_{n=1}^{\infty} E_n N_n = E$

To find such configuration we can use a mathematical trick using Lagrange multiplies: To maximize $F(x_1, x_2 ...)$ subject of constraints $f_1(x_1, x_2 ...) = 0$ and $f_2(x_1, x_2 ...) = 0$, we need to construct a function $G(x_1, x_2 ..) = F + \alpha f_1 + \beta f_2$ and set all partial derivatives equal to zero

$$\frac{\partial G}{\partial x_1} = 0 \qquad \frac{\partial G}{\partial x_2} = 0$$

Since it is inconvenient to work with ~~pro~~ products Π, we are going to maximize $\ln W(N_1, N_2 \ldots)$ so that our $G(N_1, N_2 \ldots)$ is

$$G(N_1, N_2 \ldots) = \ln W + \alpha \left[N - \sum_{n=1}^{\infty} N_n \right] + \beta \left[E - \sum_{n=1}^{\infty} E_n N_n \right]$$

By ~~calcu~~ calculating $\frac{\partial G}{\partial N_n}$ for each type of particle and setting the derivative to zero, we obtain the following functional dependence for the most probable number of particles in the n^{th} state

Distinguishable particles : $N_n = d_n e^{-(\alpha + \beta E_n)}$

Fermions : $N_n = \dfrac{d_n}{e^{(\alpha+\beta E_n)} + 1}$ Bosons: $N_n = \dfrac{d_n - 1}{e^{(\alpha+\beta E_n)} - 1}$

How do we know what are α and β? From our constrains:

$$N = \sum_{n=1}^{\infty} N_n = \sum_{n=1}^{\infty} \frac{d_n}{e^{(\alpha+\beta E_n)} \begin{pmatrix} +0 \\ +1 \\ -1 \end{pmatrix}} \begin{matrix} \text{DP} \\ \text{Fermions} \\ \text{Bosons} \end{matrix}$$

$$E = \sum_{n=1}^{\infty} E_n N_n = \sum_{n=1}^{\infty} \frac{d_n E_n}{e^{(\alpha+\beta E_n)} \begin{pmatrix} +0 \\ +1 \\ -1 \end{pmatrix}} \begin{matrix} \text{Disting. part} \\ \text{Fermions} \\ \text{Bosons} \end{matrix}$$

However, this calculation will require more specific information about energy states.

Ideal gas: a collection of non-interacting non-relativistic massive particles

Continuous spectrum, momentum $\bar{p} = (\hbar k_x, \hbar k_y, \hbar k_z)$

Energy $\to$ kinetic energy $E_k = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$

~~Each~~ Because of the uncertainty principle each state with wave-vector $\bar{k}$ occupies unique "Volume" in k space $- \pi^3/V$

(here V is a physical volume of the whole system)

Degeneracy $d_k = \frac{1}{8} \frac{4\pi k^2 dk}{(\pi^3/V)} = \frac{V}{2\pi^2} k^2 dk$ (if not include spin)

~~Are~~ Distinguishable particles: $N_n = \frac{d_n}{e^{\alpha + \beta E_n}} = e^{-\alpha} d_n e^{-\beta E_n}$

$$N = \sum_{n=1}^{\infty} N_n \to e^{-\alpha} \int_0^\infty \frac{V}{2\pi^2} k^2 dk \cdot e^{-\beta \hbar^2 k^2/2m}$$

$$N = V e^{-\alpha} \left(\frac{m}{2\pi\beta\hbar^2} \right)^{3/2}$$

$$E = \sum_{n=1}^{\infty} E_n N_n \to e^{-\alpha} \int_0^\infty \frac{V}{2\pi^2} k^2 dk \cdot \frac{\hbar^2 k^2}{2m} e^{-\beta \hbar^2 k^2/2m} = \frac{3V}{2\beta} e^{-\alpha} \left(\frac{m}{2\pi\beta\hbar^2} \right)^{3/2}$$

$$\frac{E}{N} = \frac{3}{2\beta} = \frac{3}{2} k_B T$$ from kinetic theory of an ideal monoatomic gas

Thus, it looks like $\beta = 1/k_B T$

Since we have $e^{\alpha + \beta E_n}$, it is traditional to write $\alpha = -\mu(T)/k_B T$, so that $e^{\alpha + \beta E_n} \to e^{E_n - \mu(T)/k_B T}$

$\mu(T)$ – chemical potential, the energy "price" for adding an extra particle to the system

Since $N/V = e^{-\alpha} \left(\frac{m}{2\pi\beta\hbar^2} \right)^{3/2}$,

$$\mu(T) = k_B T \left[\ln\left(\frac{N}{V}\right) + \frac{3}{2} \ln\left(\frac{2\pi\hbar^2}{m k_B T} \right) \right]$$ for the distinguishable particles

Thus, depending on statistics, we can describe an ideal gas of mass m using following distributions for the number of particles in each state $n = N/d$

Distinguishable particles!

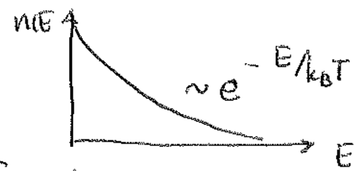
① Maxwell - Boltzmann distribution

$$n(E_k) = e^{-(E_k-\mu)/k_BT}$$

more familiar form

$$E_k = mv^2/2,$$

$$n(E_k) = e^{-mv^2/2k_BT} \cdot \frac{N}{V}\left(\frac{2\pi\hbar^2}{k_BTm}\right)^{3/2}$$



② Fermions

Fermi - Dirac distribution

$$n(E_k) = \frac{1}{e^{(E_k-\mu)/k_BT}+1} < 1$$

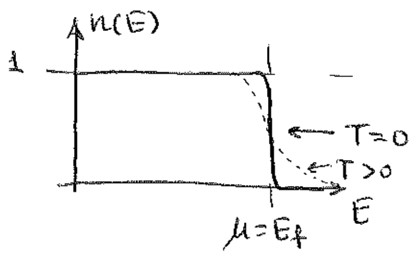
③ Bosons

~~Electr~~ Bose - Einstein distribution

$$n(E_k) = \frac{1}{e^{(E_k-\mu)/k_BT}-1}$$

Fermions at $T \to 0$

$$e^{\frac{E_k-\mu}{k_BT}} \to \begin{cases} 0 & \text{if } E_k-\mu<0 \\ \infty & \text{if } E_k-\mu>0 \end{cases}$$



$$n(E_k) \to \begin{cases} 1 & \text{if } E_k-\mu<0 \\ 0 & \text{if } E_k-\mu>0 \end{cases}$$

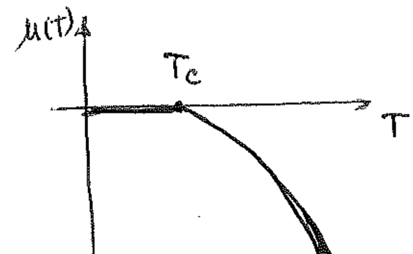
$$n(E_k) = \frac{1}{e^{(E_k-E_f)/k_BT}+1}$$

Bosons : ~~at~~ $n(E_k) \geq 0 \Rightarrow$ for $E_k=0$ $e^{-\mu/k_BT} > 1$

thus $\mu(T) < 0$

Normally, to find μ, we need to evaluate

$$N = \frac{V}{2\pi^2}\int_0^\infty \frac{k^2dk}{e^{(\hbar^2k^2/2m-\mu)/k_BT}-1}$$



$$T_c = \frac{2\pi\hbar^2}{mk_B}\left(\frac{N}{2.61\,V}\right)^{3/2}$$

For $T < T_c$ Bose - Einstein condensate is formed

Another type of boson – photon

massless $E_k = \hbar c k = \hbar\omega$, $\mu = 0$

Thermal radiation spectrum $\Rightarrow$ black body spectrum

$$N_k = \frac{d_k}{e^{E_k/k_B T} - 1} \quad , \text{ usually written for } \omega$$

$$d_k \text{ (or } d_\omega) = 2 \cdot \frac{1}{8} \frac{4\pi k^2 dk}{\pi^3/V} = \frac{V}{\pi^2} k^2 dk = \frac{V}{\pi^2 c^3} \omega^2 d\omega$$

$$dN_\omega = \frac{V}{\pi^2 c^3} \frac{\omega^2}{e^{\hbar\omega/k_B T} - 1} d\omega$$

Typically we use energy density $\rho(\omega) = \dfrac{\hbar\omega \cdot dN_\omega}{V \cdot d\omega}$

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3 \left(e^{\hbar\omega/k_B T} - 1\right)}$$