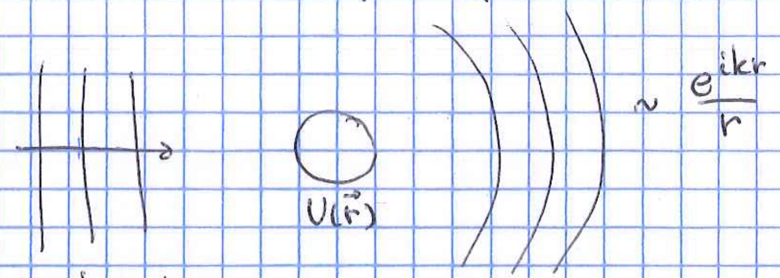
Reminder: Quantum Scattering



Incoming wave

$\sim e^{ikz}$

$$\psi(\vec r) = A\left(e^{ikz} + f(\theta)\,\frac{e^{ikr}}{r}\right)$$

For centrally-symmetric potential $U(\vec r) = U(r)$

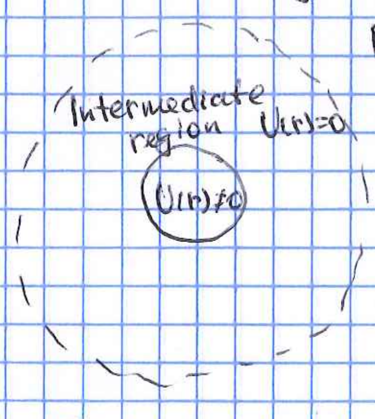
Partial wave analysis.
Since the Hamitonian commutes with angular momentum operator, the angular momentum is conserved.

Incoming wave $e^{ikz} = \sum_\ell i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\cos\theta)$

It can be presented as a sum of "spherical" waves of given angular momentum, but only $m=0$ z-component of the angular momentum.

Thus, the ~~out~~ scattered wave ~~decom~~ spherical decomposition will also only have $m=0$ components.
Moreover, each incoming wave ~~wit~~ of given ℓ must be scattered in the respective wave of the same ℓ

Three scattering regions



Radiation zone

$kr \gg 1,\ U(r) = 0$

$$\psi(r) = A\left(e^{ikz} + f(\theta)\,\frac{e^{ikr}}{r}\right)$$

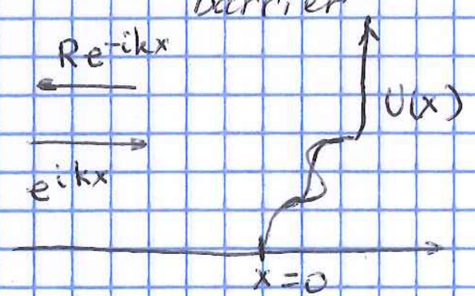
We have solved ~~the~~ for the wavefunction in the intermediate region

$$\psi(r,\theta) = A \sum_{l=0}^{\infty} i^l (2l+1) \left[j_l(kr) + ik a_l h_l^{(1)}(kr) \right] P_l(\cos\theta)$$

here a_l must be found from the boundary conditions b/w scattering region ($V(r) \neq 0$) and intermediate region,

However, we can further modify (~~specify~~ simplify) the form of partial wave amplitudes a_l

Flashback : 1D scattering from unpenetrable barrier



$$\psi(x) = A\left(e^{ikx} + Re^{-ikx}\right)$$

Flux of input particles

= flux of outgoing particles

$P_{inc} = |A|^2$

$P_{refl} = |A \cdot R|^2 = |A|^2 |R|^2$

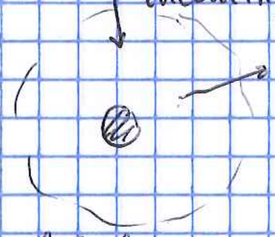
$P_{inc} = P_{refl} \Rightarrow |R|^2 = 1$

$R = e^{2i\delta}$

The scattered (reflected) wave must repeat (inverse) incoming wave, with possible phase shift

Same arguments will be valid for each partial wave (for every l)

incoming l-wave $\sim e^{-ikr}/r$



outgoing l wave $\sim e^{-ikr}/r \cdot \left(e^{2i\delta_l}\right)$

δ_l is a real number

~~so we reduced~~

Let's remove the scattering potential for a moment $U(r)=0$

$$\psi_{free}(\vec{r}) = Ae^{ikz} = A\sum_{l=0}^{\infty} i^l(2l+1)\, j_l(kr)\, P_l(\cos\theta) =$$

$$= A\sum_{l=0}^{\infty} i^l(2l+1)\frac{1}{2}\left[h_l^{(1)}(kr) + h_l^{(2)}(kr)\right] P_l(\cos\theta)$$

In the radiation zone: $kr \gg 1$

$$h_l^{(1)} \sim (-i)^{l+1}\frac{e^{ikr}}{kr} \qquad h_l^{(2)} \sim i^{l+1}\frac{e^{-ikr}}{kr}$$

outgoing wave — incoming wave

Each partial l wave

$$\psi^{(l)}_{\substack{free\\ U=0}}(\vec{r}) = \frac{A(2l+1)}{2ik}\Big[\underbrace{\frac{e^{ikr}}{r}}_{\text{outgoing}} - \underbrace{(-1)^l\, e^{-ikr}}_{\text{incoming}}\Big] P_l(\cos\theta)$$

outgoing will be phase-shifted by a potential $\to \times e^{2i\delta_l}$

For $U(r)\neq 0$ in the scattering region

$$\psi^{(l)}(\vec{r}) = \frac{A(2l+1)}{2ik}\left[\frac{1}{r}e^{ikr+2i\delta_l} - (-1)^l\frac{e^{-ikr}}{r}\right] P_l(\cos\theta)$$

Naturally, we can link δ_l with the partial wave scattering amplitude a_l

In the radiation zone $kr \gg 1$

$$\psi_k(\vec{r}) = \lim_{kr\gg 1} A\sum_{l=0}^{\infty} i^l(2l+1)\left[j_l(kr) + ika_l\, h_l^{(1)}(kr)\right] P_l(\cos\theta)$$

$$\psi^{(l)}(\vec{r}) = Ai^l(2l+1)\left[\frac{1}{2}h_l^{(1)}(kr) + \frac{1}{2}h_l^{(2)}(kr) + ika_l\, h_l^{(1)}(kr)\right] P_l(\cos\theta)$$

$$\xrightarrow{kr\gg 1} A\left\{\frac{2l+1}{2ikr}\left[e^{ikr} - (-1)^l e^{-ikr}\right] + \frac{2l+1}{r}a_l\, e^{ikr}\right\} P_l(\cos\theta)$$

$$= \frac{A(2l+1)}{2ik}\Big[\frac{e^{ikr}}{r}\underbrace{(1+2ik\,a_l)}_{e^{2i\delta_l}} - (-1)^l\frac{e^{-ikr}}{r}\Big] P_l(\cos\theta)$$

$$1 + 2ik\,a_\ell = e^{2i\delta_\ell}$$

$$a_\ell = \frac{e^{2i\delta_\ell} - 1}{2ik} = e^{i\delta_\ell}\,\frac{e^{i\delta_\ell} - e^{-i\delta_\ell}}{2ik} = e^{i\delta_\ell}\,\frac{\sin\delta_\ell}{k}$$

Why we bothered with going to δ_ℓ instead of just calculating a_ℓ?

Because a_ℓ is a complex number $\rightarrow$ two values

δ_ℓ is a real number $\rightarrow$ one value, we cut the amount of calculations in half

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell\, P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{i\delta_\ell}\,\frac{\sin\delta_\ell}{k}\, P_\ell(\cos\theta)$$

$$\sigma = \sum_{\ell=0}^{\infty} 4\pi\,(2\ell+1)\,|a_\ell|^2 = \sum_{\ell=0}^{\infty} 4\pi\,(2\ell+1)\,\frac{\sin^2\delta_\ell}{k^2}$$

Partial wave scattering cross-sections

$$\sigma_\ell = 4\pi(2\ell+1)\frac{\sin^2\delta_\ell}{k^2} \qquad \sigma = \sum_{\ell=0}^{\infty} \sigma_\ell$$

You will hear people talking about

s - scattering $(\ell = 0)$ σ_0 or σ_s or

p - scattering $(\ell = 1)$ σ_1 or σ_p etc.