Physics 313 Midterm test #2
November 8, 2023

Name (please print): Solutions

This test is administered under the rules and regulations of the honor system of the College of William & Mary.

Signature: ______________________________

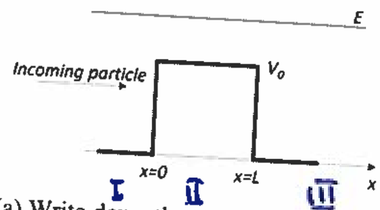
Final score: ___________

Show all work to receive credit, and circle your final answers. This exam is closed book, but you can use a prepared index card with reference information that you have prepared.

Problem 3 (30 points)

A particle with mass m and total energy E approaches the potential barrier $V(x)$from the left, as shown: $V(x) = \begin{cases} 0 & x \le 0 \\ V_0 & 0 < x < L \\ 0 & x \ge L \end{cases}$

The height of the barrier is less than the particle total energy $V_0 < E$.



$$k_1 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

(a) Write down the general expression for the particle wave function $\psi(x)$ for all value of x.
(b) State the boundary conditions for $x = 0, L$.
(c) Assuming that $E = \dfrac{\pi^2\hbar^2}{2mL^2}$ and $V_0 = \dfrac{3\pi^2\hbar^2}{8mL^2}$, calculate the reflection coefficient R of the particle from this barrier.

a) $$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x<0 \\ Ce^{ik_1x} + De^{-ik_1x} & 0<x<L \\ Fe^{ik(x-L)} & x>L \end{cases}$$

b) Boundary conditions

$x=0$: $\psi_I(0) = \psi_{II}(0)$ $\quad A+B = C+D$

$\psi_I'(0) = \psi_{II}'(0)$ $\quad ik(A-B) = ik_1(C-D)$

$x = L$ $\quad \psi_{II}(L) = \psi_{III}(L)$ $\quad Ce^{ik_1L} + De^{-ik_1L} = F$

$\psi_{II}'(L) = \psi_{III}'(L)$ $\quad ik_1Ce^{ik_1L} - ik_1De^{-ik_1L} = ikF$

c) $$k = \sqrt{\frac{2m}{\hbar^2}\cdot\frac{\pi^2\hbar^2}{2mL^2}} = \frac{\pi}{L} \qquad k_1 = \sqrt{\frac{2m}{\hbar^2}\,\frac{\pi^2\hbar^2}{8mL^2}} = \frac{\pi}{2L}$$

$$e^{ik_1L} = e^{i\pi/2} = i \qquad e^{-ik_1L} = e^{-i\pi/2} = -i$$

$iC - iD = F \quad \Rightarrow C - D = -iF$

$ik_1(iC) - ik_1(-iD) = -k_1(C+D) = ikF$

$C+D = -i\frac{k}{k_1}F = -2iF$

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$A+B = -2iF$

$ik(A-B) = ik_1(-iF) = k_1 \cdot F = \frac{k_1}{2}i(A+B)$

$A - B = \frac{1}{4}(A+B) \qquad B = \frac{3}{5}A \qquad R = \frac{|B|^2}{\ldots} = 16$

Problem 2 (30 points)
A particle with intrinsic spin one is placed in a constant external magnetic field B_0 in the y direction. The initial spin state of the particle is $|1,1\rangle$, that is, a state with $S_z = \hbar$. The Hamiltonian of the particle in the magnetic field is $\hat{H} = \omega_0 \hat{S}_y$.
Determine the probability that the particle is in the state $|1,-1\rangle$ at time t.

The eigenstates of the operator $\hat{S}_y$ in terms of the eigenstates of $\hat{S}_z$ are:

$$|S_y = \hbar\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}, \ |S_y = 0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ |S_y = -\hbar\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix}.$$

To find the time evolution, we need to decompose the initial state in $\hat{S}_y$ basis

$$\langle S_y = \hbar | 1,1\rangle = \frac{1}{2}(1 \ \ -i\sqrt{2} \ \ -1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}; \quad \langle S_y = -\hbar | 1,1\rangle = \frac{1}{2}(1 \ \ i\sqrt{2} \ \ -1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$\langle S_y = 0 | 1,1\rangle = \frac{1}{\sqrt{2}}(1 \ \ 0 \ \ 1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$|d(0)\rangle = |1,1\rangle = \frac{1}{2}|S_y = \hbar\rangle + \frac{1}{2}|S_y = -\hbar\rangle + \frac{1}{\sqrt{2}}|S_y = 0\rangle$$

$\downarrow$ time evolution

$$|d(t)\rangle = \frac{1}{2}e^{-i\omega_0 t}|S_y = \hbar\rangle + \frac{1}{2}e^{i\omega_0 t}|S_y = -\hbar\rangle + \frac{1}{\sqrt{2}}|S_y = 0\rangle$$

$$\langle 1,-1 | d(t)\rangle = \frac{1}{2}e^{-i\omega_0 t}\langle 1,-1 | S_y = \hbar\rangle + \frac{1}{2}e^{i\omega_0 t}\langle 1,-1 | S_y = -\hbar\rangle + \frac{1}{\sqrt{2}}\langle 1,-1 | S_y = 0\rangle$$

$$= \frac{1}{2}e^{-i\omega_0 t}\left(-\frac{1}{2}\right) + \frac{1}{2}e^{i\omega_0 t}\left(-\frac{1}{2}\right) + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} =$$

$$= \frac{1}{2} - \frac{1}{4}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) = \frac{1}{2} - \frac{1}{2}\cos\omega_0 t = \sin^2\frac{\omega_0 t}{2}$$

$$P_{1,-1} = |\langle 1,-1 | d(t)\rangle|^2 = \sin^4\frac{\omega_0 t}{2}$$

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Potentially useful information

Spin-1/2 particle

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\ \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\ \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

Eigenstates for the spin operators:

$$|+z\rangle = \begin{pmatrix}1\\0\end{pmatrix};\ |-z\rangle = \begin{pmatrix}0\\1\end{pmatrix};\ |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix};\ |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix};\ |+y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix};\ |-y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

Spin-1 particle

$$\hat{S}_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\ \hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\ \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}.$$

Eigenstates of the $\hat{S}_z$ operator (in the z-basis):

$$|1,1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix};\ |1,0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix};\ |1,-1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

The commutator of two operators is defined as $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

Dirac delta function

$$\int_a^b \delta(x-x_0)dx = \begin{cases}1 & a \le x_0 \le b\\ 0 & \text{otherwise}\end{cases}$$

$$\int_a^b \delta(x-x_0)f(x)dx = \begin{cases}f(x_0) & a \le x_0 \le b\\ 0 & \text{otherwise}\end{cases}$$

Kronecker delta symbol:

$$\delta_{nk} = \begin{cases}1 & n = k\\ 0 & n \ne k\end{cases}$$

Differential equations:

$\frac{d^2y}{dx^2} = -k^2 y$, possible solutions $y_{1,2} = \sin(kx)$ and $\cos(kx)$ or $y_{1,2} = e^{\pm ikx}$

$\frac{d^2y}{dx^2} = \kappa^2 y$, possible solutions $y_{1,2} = e^{\pm\kappa x}$

Orthogonality of the trigonometric functions:

$\int_0^L \sin\frac{\pi n x}{L}\sin\frac{\pi k x}{L}dx = \frac{L}{2}\delta_{nk}$,

$\int_0^L \cos\frac{\pi n x}{L}\cos\frac{\pi k x}{L}dx = \frac{L}{2}\delta_{nk}$,

$\int_0^L \sin\frac{\pi n x}{L}\cos\frac{\pi k x}{L}dx = 0$

Potentially useful mathematical expressions

$$i\cdot i = -1;\ i\cdot(-i) = 1;\ 1/i = -i;$$
$$e^{i\phi} = \cos\phi + i\sin\phi;\ \cos\phi = (e^{i\phi}+e^{-i\phi})/2;\ \sin\phi = (e^{i\phi}-e^{-i\phi})/2i;$$
$$|e^{i\phi}|^2 = 1;$$
$$\cos 2\phi = \cos^2\phi - \sin^2\phi;\ \sin 2\phi = 2\sin\phi\cos\phi$$

Problem 1(40 points)

A particle of mass m is trapped inside the infinite square well potential well $V(x) = \begin{cases} 0 & -a \le x \le a \\ \infty & \text{elsewhere} \end{cases}$

(a) Verify that the wave functions

$$\psi_A(x) = \begin{cases} \sqrt{\frac{1}{a}} \cos\left(\frac{\pi x}{2a}\right) & -a \le x \le a \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad \psi_B(x) = \begin{cases} \sqrt{\frac{1}{a}} \cos\left(\frac{3\pi x}{2a}\right) & -a \le x \le a \\ 0 & \text{elsewhere} \end{cases}$$

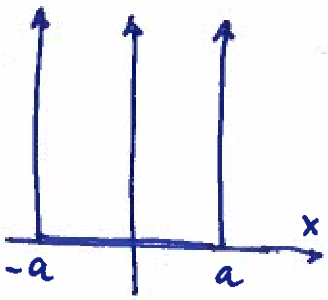
are the eigenfunctions of this potential, and find the corresponding energy eigenvalues.

For the rest of the problem assume that the initial state of the particle is $\psi(x) = \frac{3i}{5}\psi_A(x) - \frac{4}{5}\psi_B$.

(b) Write the time evolution of this state $\psi(x,t)$.

(c) What is the average energy $\langle E \rangle$ of the particle in this state?

(d) Write the expression to calculate the square of the average position $\langle x^2(t) \rangle$ of the particle in this state. Without evaluating the integral give an argument that it is time-dependent.



a) Schrodinger equation $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$

ψ_A: $\left(-\frac{\hbar^2}{2m}\right)\left(-\frac{\pi^2}{4a^2}\right)\frac{1}{\sqrt{a}}\cos\frac{\pi x}{2a} = E_A \frac{1}{\sqrt{a}}\cos\frac{\pi x}{2a}$; $\underline{E_A = \frac{\pi^2\hbar^2}{8ma^2}}$

ψ_B: $\left(-\frac{\hbar^2}{2m}\right)\left(-\frac{9\pi^2}{4a^2}\right)\frac{1}{\sqrt{a}}\cos\frac{3\pi x}{2a} = E_B \cdot \frac{1}{\sqrt{a}}\cos\frac{3\pi x}{2a}$; $\underline{E_B = \frac{9\pi^2\hbar^2}{8ma^2}}$

b) $\psi(x,t) = \frac{3i}{5}\psi_A(x)e^{-iE_At/\hbar} - \frac{4}{5}\psi_B(x)e^{-iE_Bt/\hbar}$

c)

$\langle E \rangle = E_A \cdot \left(\frac{3}{5}\right)^2 + E_B\left(\frac{4}{5}\right)^2 = \left(\frac{9}{25} + \frac{16 \cdot 9}{25}\right)\frac{\pi^2\hbar^2}{8ma^2} = \frac{153}{200}\frac{\pi^2\hbar^2}{ma^2}$

d) $\langle x^2(t) \rangle = \int_{-a}^{a} \psi^*(x,t) \cdot x^2 \psi(x,t)\, dx = \int_{-a}^{a} |\psi(x,t)|^2 \cdot x^2 dx$

$|\psi(x,t)|^2 = \left(\frac{3i}{5}\psi_A(x)e^{-iE_At/\hbar} - \frac{4}{5}\psi_B(x)e^{-iE_Bt/\hbar}\right) \times$

$\times\left(-\frac{3i}{5}\psi_A(x)e^{iE_At/\hbar} - \frac{4}{5}\psi_B(x)e^{iE_Bt/\hbar}\right) =$

$= \frac{9}{25}\psi_A^2(x) + \frac{16}{25}\psi_B^2(x) - \frac{12i}{25}\psi_A\psi_B e^{i(E_B-E_A)t/\hbar} + \frac{12i}{25}\psi_A\psi_B e^{-i(E_B-E_A)t/\hbar}$

$= \frac{9}{25}\psi_A^2(x) + \frac{16}{25}\psi_B^2(x) + \frac{6}{25}\psi_A(x)\psi_B(x)\sin(E_B-E_A)t/\hbar$

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Since $\int_{-a}^{a}\psi_A(x)\psi_B(x)\cdot x^2 dx \neq 0$ then there will be the time-dependent term in $\langle x^2 \rangle$