Position and momentum operators

quantum state
(contains all information)

interested in magnetic properties
↓
interested in spin (angular momentum)
↓
describe states using basis of $\hat{S}^2$, $\hat{S}_z$ operators
discrete variables
e.g., $|\pm z\rangle$, $|s=1, m=0\rangle$

interested in time evolution
↓
interested in energy
↓
describe states using basis of Hamiltonian stationary states
$|E\rangle$

interested in motion/spatial description
↓
interested in position and momentum
↓
describe states using basis of position operator $\hat{x}$
$\psi_\alpha(x)$

Position operator $\hat{x}$
position operator eigenstates $|x\rangle$

$$\hat{x}|x\rangle = \underbrace{x}_{\text{position value}}|x\rangle$$

Momentum operator $\hat{p}_x$
momentum operator eigenstates

$$\hat{p}_x|p\rangle = \underbrace{p}_{\text{momentum value}}|p\rangle$$

Position and momentum operators <u>do not commute</u>

$$[\hat{x}, \hat{p}_x] = i\hbar \quad \Rightarrow \quad \Delta x \cdot \Delta p \geq \hbar/2$$

~~can~~ no joint eigen~~fu~~states
impossible for a particle to have defined position

Eigenstate of the position $\hat{x}|x\rangle = x|x\rangle$ $\quad \Delta x = 0$

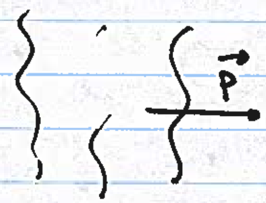
no information about p value

x is known

localize particle $\rightarrow$ cannot know where it is going

Eigenstate of the momentum $\hat{p}_x|p\rangle = p|p\rangle$ $\quad \Delta p = 0$

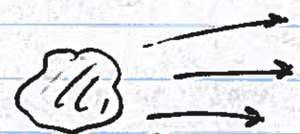
no information of position



plane wave

completely delocalize

More realistic case some uncertainty in both momentum and position



some information about future motion

some localization

In most situation we are going to work in position representation, although it is certainly possible to use the momentum representation as well

$\{|x\rangle\}$ vs $\{|p\rangle\}$

$$|\alpha\rangle = \int_{-\infty}^{+\infty} \langle x|\alpha\rangle |x\rangle dx \qquad |\alpha\rangle = \int_{-\infty}^{+\infty} \langle p|\alpha\rangle |p\rangle dp$$

$$= \int_{-\infty}^{+\infty} \psi_\alpha(x) |x\rangle dx \qquad = \int_{-\infty}^{+\infty} \varphi_\alpha(p) |p\rangle dp$$

$\varphi_\alpha(p) = \langle p|\alpha\rangle$

$\psi_\alpha(x) = \langle x|\alpha\rangle$ — a wave function

(this basis is commonly used for scattering problems)

An operator average

$$\langle \hat{A}\rangle = \langle \alpha|\hat{A}|\alpha\rangle = \int \psi_\alpha^* \hat{A} \psi_\alpha dx$$

WHEN PEOPLE TALK ABOUT THE SAME THING IN DIFFERENT LANGUAGES

$$|\psi\rangle = \int d^3r\, |\vec{r}\rangle\langle\vec{r}|\psi\rangle = \int d^3r\, \langle\vec{r}|\psi\rangle\, |\vec{r}\rangle$$

$$|\psi\rangle = \int d^3p\, |\vec{p}\rangle\langle\vec{p}|\psi\rangle = \int d^3p\, \langle\vec{p}|\psi\rangle\, |\vec{p}\rangle$$

Momentum operator in x- basis

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$$

initial state: $|\alpha\rangle \longrightarrow \psi_\alpha(x)$

final state: $|\beta\rangle \longrightarrow \psi_\beta(x)$

$$\psi_\beta(x) = \hat{p}_x \psi_\alpha(x) = -i\hbar \frac{\partial \psi_\alpha(x)}{\partial x}$$

Eigen state of the momentum operator

$$\hat{p}_x |p\rangle = p|p\rangle \qquad \psi_p(x) = \langle x|p\rangle$$

$$\hat{p}_x \psi_p(x) = p\psi_p(x) \implies -i\hbar \frac{\partial \psi_p(x)}{\partial x} = p\psi_p(x)$$

$$\frac{\partial \psi_p(x)}{\partial x} = \frac{ip}{\hbar}\psi_p(x)$$

$$\psi_p(x) \propto \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

normalization

$\longrightarrow$ describes a plane wave moving with momentum p in $+x$ direction

$$\psi_p(x) \propto \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}$$

$\longrightarrow$ describes a plane wave moving with momentum p in $-x$ direction

These states are great to describe the time evolution of free particles

$$\hat{H} = \frac{\hat{p}^2}{2m} \qquad \hat{H}|p\rangle = \frac{p^2}{2m}|p\rangle$$

$$|p\rangle \longrightarrow |p\rangle e^{-i\left(\frac{p^2}{2m}\right)\frac{t}{\hbar}}$$

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar - i\,p^2/2m\hbar \cdot t}$$

same as complex representation of any wave!

Constant potential

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{U}_0 \qquad U_0 = \text{const}$$

$$\hat{H}|p\rangle = \left(\frac{\hat{p}^2}{2m} + \hat{U}_0\right)|p\rangle = \underbrace{\left(\frac{p^2}{2m} + U_0\right)}_{E} |p\rangle$$

$$E = \frac{p^2}{2m} + U_0 \iff p = \sqrt{(E-U_0)\cdot 2}$$

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar - iEt/\hbar}$$ motion along x

$$\psi_{-p}(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{-ipx/\hbar - iEt/\hbar}$$ motion against x

Average momentum in a state $|p\rangle$

$$\langle p\rangle = \langle p|\hat{p}|p\rangle = p\langle p|p\rangle = p$$ as expected

$\Delta p = 0$ no uncertainty

On the other hand, the probability to find a particle b/w $a < x < b$

$$P_{ab} = \int_a^b |\psi_p(x)|^2\, dx = \frac{1}{2\pi\hbar}(b-a)$$ $\rightarrow$ equally possible every where

the particle is completely delocalized.