Orthogonality of the continuous state

$\hat{x}|x\rangle = x|x\rangle$ How to do normalization

$$|\alpha\rangle = \int_{-\infty}^{+\infty} \langle x|\alpha\rangle \, |x\rangle \, dx$$

if $|\alpha\rangle = |x'\rangle$ $$|x'\rangle = \int_{-\infty}^{+\infty} \underbrace{\langle x|x'\rangle}_{\text{function of } x,\, x'} \cdot |x\rangle \, dx$$

Dirac detra function $\delta(x-x')$

$$\delta(x-x') = \begin{cases} 0 & x \neq x' \\ \infty & x = x' \end{cases} \quad \text{such that} \quad \int_{-\infty}^{+\infty} \delta(x-x')\,dx = 1$$

infinitely tall infinitely skinny peak

Why you should love it!

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-x')\,dx = f(x')$$

$$\int_{-\infty}^{+\infty} |x\rangle\,\delta(x-x')\,dx = |x'\rangle$$

Momentum space: $|p\rangle \rightarrow \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

$\langle p|p'\rangle = \delta(p-p')$

$$\langle p|p'\rangle = \int_{-\infty}^{+\infty} \langle p|x\rangle\langle x|p'\rangle\,dx = \left(\frac{1}{\sqrt{2\pi\hbar}}\right)^2 \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\, e^{ip'x/\hbar}\,dx$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{i\frac{(p'-p)x}{\hbar}}\,dx$$

One version of δ-function

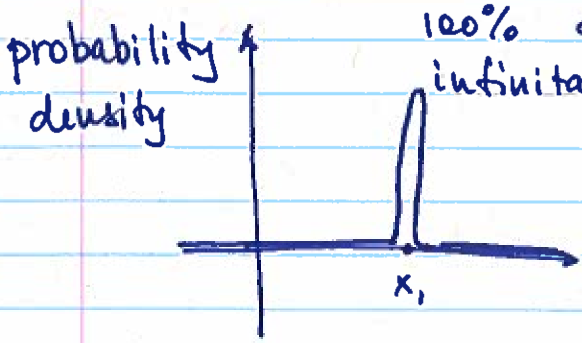
$$\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(x-x')t}\,dt$$

Such strange δ-function value actually make sense...

x-operator eigenstate

~~Probability density~~ 0% chance to find particle outside of $x = x'$

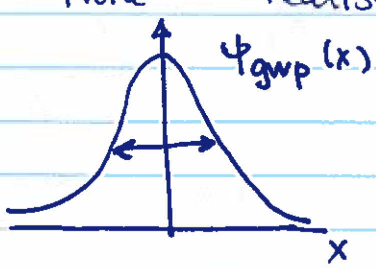
100% chance to find it within any infinitasmall region around x'



p-operator eigenstate — ~~states~~ plane wave completely delocolized, but having precize wavelength $\lambda = 2\pi\hbar/p$,

again, no chance to measure particle's momentum any value except p

More realistic case → Gaussian wave packet



we kinda know position and momentum, but not accurately

$$\psi_{gwp}(x) = \langle x | gwp \rangle = \frac{1}{\pi^{1/4} a^{1/2}} e^{-x^2/2a^2}$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} x \cdot |\psi_{gwp}(x)|^2 dx = 0$$

Uncertainty $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \langle x \rangle^2 =$

$$\int_{-\infty}^{+\infty} x^2 |\psi_{gwp}(x)|^2 dx = \frac{1}{\sqrt{\pi} a} \int_{-\infty}^{+\infty} x^2 e^{-x^2/\cancel{2}a^2} dx = \frac{a^2}{2}$$

$$\langle p_x \rangle = \langle gwp | \hat{p}_x | gwp \rangle = \frac{\hbar}{i} \int_{-\infty}^{+\infty} \psi^*(x) \frac{\partial}{\partial x} \psi(x)\, dx =$$

$$= \frac{\hbar}{i} \frac{1}{\sqrt{\pi} a} \int_{-\infty}^{+\infty} e^{-x^2/2a^2} \frac{\partial}{\partial x}\left(e^{-x^2/2a^2}\right) dx = \frac{\hbar}{i} \frac{1}{\sqrt{\pi}\, a} \left(-\frac{1}{a^2}\right) \times$$

$$\times \int_{-\infty}^{+\infty} x\, e^{-x^2/a^2}\, dx = 0$$

$$\langle p_x^2 \rangle = \langle gwp | \hat{p}_x^2 | gwp \rangle = -\hbar^2 \int \psi^*(x) \frac{\partial^2}{\partial x^2} \psi(x)\, dx =$$

$$= -\hbar^2 \frac{1}{\sqrt{\pi} a} \int_{-\infty}^{+\infty} e^{-x^2/2a^2} \frac{\partial^2}{\partial x^2} \left(e^{-x^2/2a^2}\right) dx =$$

$$= -\hbar^2 \frac{1}{\sqrt{\pi} a} \int_{-\infty}^{+\infty} e^{-x^2/a^2} \left[\left(-\frac{1}{2a^2}\right)^2 x^2 e^{-x^2/2a^2} + \left(-\frac{1}{2a}\right) e^{-x^2/2a^2}\right] dx$$

$$= \hbar^2 / 2a^2$$

$$\Delta p^2 = \langle p_x^2 \rangle - \langle p_x \rangle^2 = \hbar^2 / 2a^2$$

$$\Delta x \cdot \Delta p = \frac{a}{\sqrt{2}} \cdot \frac{\hbar}{\sqrt{2} a} = \frac{\hbar}{2}$$

$$\langle p | gwp \rangle = \int_{-\infty}^{+\infty} dx\, \langle p | x \rangle \langle x | gwp \rangle =$$

$$= \int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \frac{1}{\sqrt{\sqrt{\pi} a}} e^{-x^2/2a^2} =$$

$$= \int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{\sqrt{\pi}\, a}} e^{-\frac{1}{2a^2}\left(x + \frac{ipa^2}{\hbar}\right)^2} \cdot e^{-\frac{p^2 a^2}{2\hbar^2}} =$$

$$= \sqrt{\frac{a}{\hbar \sqrt{\pi}}}\; e^{-p^2 a^2 / 2\hbar^2}$$

A gaussian wave packet has the same basic shape both in the coordinate and momentum representations (bases)

Stationary Gaussian wave packet

$\underline{\underline{t=0}}$ $\quad \psi(x) = \frac{1}{\sqrt{\sqrt{\pi}a}} e^{-x^2/2a^2} = \langle x|gwp\rangle$

$$\langle p|gwp\rangle = \sqrt{\frac{a}{\sqrt{\pi}\hbar}}\, e^{-p^2a^2/2\hbar^2}$$

Time evolution $\quad |gwp(t>0)\rangle = \int_{-\infty}^{+\infty} dp\, |p\rangle\langle p|gwp(t=0)\rangle$

$$|gwp(t)\rangle = e^{-i\hat{H}t/\hbar}|gwp(t=0)\rangle = \int_{-\infty}^{+\infty}\left[e^{-i\hat{H}t/\hbar}|p\rangle\right]\langle p|gwp(t=0)\rangle =$$

$$= \int_{-\infty}^{+\infty} e^{-ip^2t/2m\hbar}\,|p\rangle\langle p|gwp(t=0)\rangle$$

In the momentum space $\quad \langle p'|gwp(t)\rangle = \int_{-\infty}^{+\infty} e^{-ip^2t/2m\hbar}\,\underbrace{\langle p'|p\rangle}_{\delta(p'-p)} \times \langle p|gwp(t=0)\rangle$

$$\langle p'|gwp(t)\rangle = \underbrace{e^{-ip'^2t/2m\hbar}}_{\text{only a phase factor!}}\langle p'|gwp(t=0)\rangle$$

$$|\langle p'|gwp(t)\rangle|^2 = |\langle p'|gwp(t=0)\rangle|^2$$

probability distribution in the momentum space does not change in time!

However, ~~$\psi(x,t)$~~

$$\psi(x,t) = \langle x|gwp(t)\rangle = \int_{-\infty}^{+\infty} dp\, e^{-ip^2t/2m\hbar}\,\langle x|p\rangle\,\langle p|gwp(t=0)\rangle =$$

$$= \int_{-\infty}^{+\infty} dp\, e^{-ip^2t/2m\hbar}\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}\sqrt{\frac{a}{\hbar\sqrt{\pi}}}\,e^{-p^2a^2/2\hbar^2} =$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{a}{\hbar\sqrt{\pi}}}\int_{-\infty}^{+\infty} dp\, e^{-\left[\frac{a^2}{2\hbar}+i\frac{t}{2m\hbar}\right]p^2+ipx} =$$

$$= \frac{1}{\sqrt{\sqrt{\pi}\,(a+i\hbar t/ma)}}\, e^{-x^2/2a(1+i\hbar t/ma^2)}$$

Probability density

$$|\psi(x)|^2 = \frac{1}{\sqrt{\pi}\,(a^2 + \hbar^2 t^2/m^2 a^2)}\, e^{-x^2/a^2(1+\hbar^2 t^2/(ma^2)^2)}$$

In the ~~exact~~ position basis the wavepacket spreads out!

$$\Delta x = \frac{a}{\sqrt{2}} \sqrt{1 + \hbar^2 t^2/(ma^2)^2}$$

What if the wavepacket is not stationary, but has some average momentum $\langle p \rangle = p_0$?

$$\langle p | gwp \rangle = \sqrt{\frac{a}{\hbar\sqrt{\pi}}}\, e^{-(p-p_0)^2 a^2/2\hbar^2}$$

and $\psi(x,t) = \frac{1}{\pi^{1/4}\sqrt{a}}\, e^{-x^2/2a^2}\, e^{ip_0 x/\hbar}$

~~(we can always move to the reference frame in which the wavepacket is stationary)~~

The non-zero momentum will manifest itself as a complex phase on top of the original wave packet, even though its initial ~~d~~ probability distribution stays the same

$$\left|\psi(x,t)\big|_{p=0}\right|^2 = \left|\psi(x,t)\big|_{p=p_0}\right|^2 = \frac{1}{\sqrt{\pi}\,a}\, e^{-x^2/a^2}$$

x- representation is the ultimate "particle"
p- representation is the ultimate "wave"
of a wave-particle quantum duality

Not surprisingly, wrighting the state wave function in p-representation is equivalent to taking the fourier transform of $\psi_\alpha(x)$

$$|\alpha\rangle \longrightarrow \psi_\alpha(x) = \langle x|\alpha\rangle \quad \text{(x-representation)}$$

$$\varphi(p) = \langle p|\alpha\rangle = \int_{-\infty}^{+\infty} \langle p|x\rangle\langle x|\alpha\rangle \, dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar} \, \psi(x) \, dx$$

Same form as $F(\omega) = \frac{1}{\sqrt{2\pi}} \int e^{-i\omega t} f(t) dt$