To commute or not to commute?
There is some uncertainty

Definition: a commutator of two operators is defined as

$$[\hat{A},\hat{B}]|\alpha\rangle \doteq \hat{A}\hat{B}|\alpha\rangle - \hat{B}\hat{A}|\alpha\rangle = -[\hat{B},\hat{A}]|\alpha\rangle$$

Two operators commute if for <u>any</u> $|\alpha\rangle$

$$\hat{A}\hat{B}|\alpha\rangle = \hat{B}\hat{A}|\alpha\rangle$$

i.e. their order does not matter

If operators commute, they will have a basis of <u>common</u> <u>eigenstates</u>

Example of a commuting operators:

- $\hat{J}_z$ and $\hat{R}(\varphi\vec{k}) = \exp\left(-\frac{i\varphi}{\hbar}\hat{J}_z\right)$
- $\hat{J}_z$ and $\hat{P}_\pm$

A unity operator $\hat{\mathbb{1}}$ commutes with any other operators

We can easily check that the operators commute using their matrix form

$$\hat{J}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{R}(\varphi\vec{k}) = \begin{pmatrix} \langle +z|e^{-\frac{i\varphi}{\hbar}\hat{J}_z}|+z\rangle & \langle +z|e^{-\frac{i\varphi}{\hbar}\hat{J}_z}|-z\rangle \\ \langle +z|e^{-\frac{i\varphi}{\hbar}J_z}|+z\rangle & \langle -z|e^{-\frac{i\varphi}{\hbar}J_z}|-z\rangle \end{pmatrix}$$

$$e^{-\frac{i\varphi}{\hbar}\hat{J}_z}|\pm z\rangle = e^{-\frac{i\varphi}{\hbar}\left(\pm\frac{\hbar}{2}\right)}|\pm z\rangle = e^{\mp i\varphi/2}|\pm z\rangle$$

$$\hat{R}(\varphi\vec{k}) = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\hat{J}_z}\underbrace{\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}}_{\hat{R}(\varphi\vec{k})} = \underbrace{\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}}_{\hat{R}(\varphi\vec{k})}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\hat{J}_z} = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & -e^{i\varphi/2} \end{pmatrix}$$

Non-commuting operators $\hat{A}\hat{B}|\alpha\rangle \neq \hat{B}\hat{A}|\alpha\rangle$

Examples of non-commuting operators

- Different components of the angular momentum

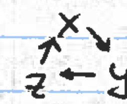
$$[\hat{J}_x, \hat{J}_y] \neq 0 \qquad [\hat{J}_x, \hat{J}_z] \neq 0 \qquad [\hat{J}_y, \hat{J}_z] \neq 0$$

- Rotation operators around different axes do not commute
- Pauli matrices do not commute

$$\hat{\sigma}_x \hat{\sigma}_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\hat{\sigma}_z$$

$$\hat{\sigma}_y \hat{\sigma}_x = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\hat{\sigma}_z$$

$$[\hat{\sigma}_x, \hat{\sigma}_y] = \hat{\sigma}_x\hat{\sigma}_y - \hat{\sigma}_y\hat{\sigma}_x = 2i\hat{\sigma}_z$$



One can check: $[\hat{\sigma}_y, \hat{\sigma}_z] = 2i\hat{\sigma}_x$

$$[\hat{\sigma}_z, \hat{\sigma}_x] = 2i\hat{\sigma}_y$$

What is the significance of having non-commuting operators?

They cannot be precisely measured at the same time (or at the same state)

Uncertainty principle

~~If~~ $[\hat{A}, \hat{B}] = i\hat{C}$ that $\Delta A \cdot \Delta B \geq \frac{1}{2}|\langle\hat{C}\rangle|$

where ΔA and ΔB are uncertainties of the operators $\hat{A}$ & $\hat{B}$

$$\Delta A = \sqrt{\langle(\hat{A} - \langle A\rangle)^2\rangle} \qquad \Delta B = \sqrt{\langle(\hat{B} - \langle \hat{B}\rangle)^2\rangle}$$

$\langle C\rangle$ = the expectation value of $\hat{C}$

Since the components of the angular momentum do not commute, we cannot measure them simultaneously

$$\hat{J}_x = \frac{\hbar}{2}\hat{\sigma}_x \qquad \hat{J}_y = \frac{\hbar}{2}\hat{\sigma}_y \qquad [\hat{J}_x, \hat{J}_y] = \frac{\hbar^2}{4}[\sigma_x, \sigma_y] =$$

$$= \frac{\hbar^2}{4}\cdot 2i\,\sigma_z = i\hbar\,\hat{J}_z$$

$$[\hat{J}_x, \hat{J}_y] = i\hbar\,\hat{J}_z$$

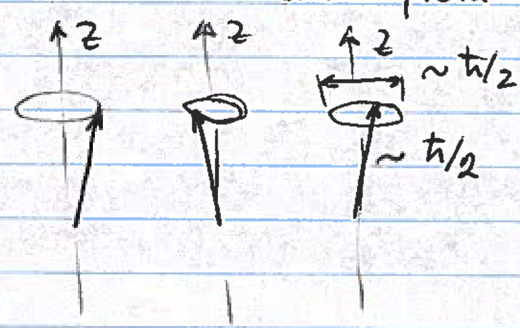
If we are in $|\pm z\rangle$ $\quad \langle \pm\hat{J}_z\rangle = \pm\frac{\hbar}{2}$

$$\Delta J_x \cdot \Delta J_y \geqslant \frac{1}{2}\cdot\hbar\left|\pm\frac{\hbar}{2}\right| = \frac{\hbar^2}{4}$$

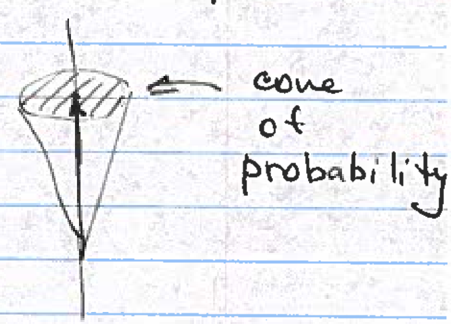
$$\underbrace{\Delta J_x = \Delta J_y}_{\text{due to symmetry}} \geqslant \hbar/2$$

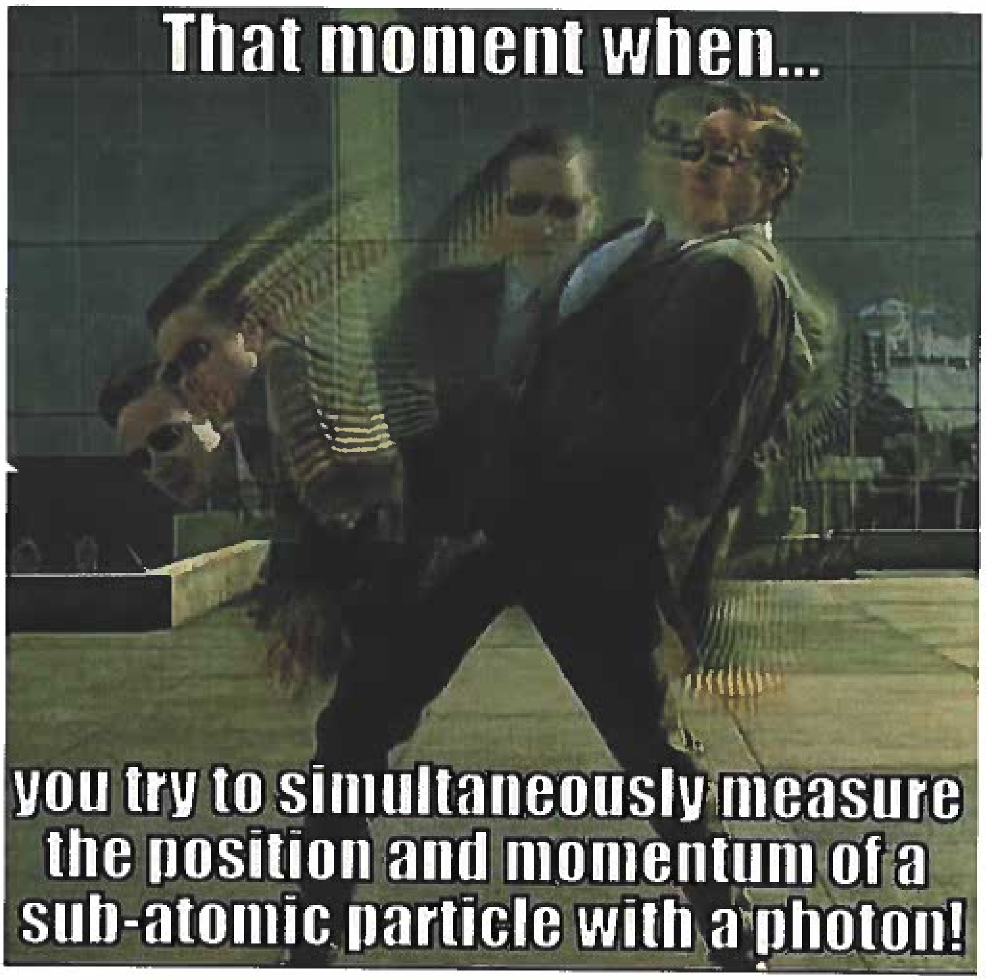
So even if spins are perfectly aligned (on average) along z, their transverse component cannot be exactly zero

Semi-classical picture



Quantum picture




That moment when...
you try to simultaneously measure the position and momentum of a sub-atomic particle with a photon!