Time evolution of a quantum state

$$|\alpha(t=0)\rangle \xrightarrow[\text{time-evolution operator}]{\hat{U}(t)} |\alpha(t)\rangle = \hat{U}(t)\,|\alpha(t=0)\rangle$$

Time evolution operator (unitary) : $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$

Hamiltonian $\hat{H}$ is the energy operator

Shrodinger equation: $i\hbar \frac{\partial}{\partial t}|\alpha(t)\rangle = \hat{H}|\alpha(t)\rangle$

$\hat{H} = \hat{H}^{\dagger}$ hermitian operator (has to have real eigenvalues)

$$|\alpha(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\alpha(0)\rangle$$

Time evolution is trivial for the eigenstates of the Hamiltonian

$\hat{H}|E\rangle = E|E\rangle$ — E: eigenvalues, $|E\rangle$: eigenstates — may be finite #, may be infinite # of specific states, may be a continuum

If $|E\rangle = |E(0)\rangle$ is the eigenstate of $\hat{H}$:

$$\boxed{\begin{aligned}&\hat{H}|E(0)\rangle = E|E(0)\rangle\\ &\hat{H}^2|E(0)\rangle = E^2|E(0)\rangle\\ &\dots\end{aligned}}$$

$$|E(t)\rangle = e^{-i\hat{H}t/\hbar}|E(0)\rangle = \left[\mathbb{1} - \frac{i\hat{H}t}{\hbar}\hat{H} + \left(\frac{it}{\hbar}\right)^2\hat{H}^2 + \dots\right]|E(0)\rangle$$

$$= \left[|E(0)\rangle - \frac{iEt}{\hbar}|E(0)\rangle + \left(\frac{itE}{\hbar}\right)^2|E_0\rangle\right] = e^{-iEt/\hbar}\,|E(0)\rangle$$

Time evolution only adds a complex phase so that any observables calculated in this state will stay unchanged

Two-level system

state 2 —— E_2

$\Delta E = E_2 - E_1$

state 1 —— E_1

$$\hat{H}|1\rangle = E_1|1\rangle$$
$$\hat{H}|2\rangle = E_2|2\rangle$$

If $|\alpha(t=0)\rangle = |1\rangle \Rightarrow |\alpha(t)\rangle = e^{-\frac{iE_1 t}{\hbar}}\underbrace{|1\rangle}_{|\alpha(t=0)\rangle}$

If $|\alpha(t=0)\rangle = |2\rangle \Rightarrow |\alpha(t)\rangle = e^{-iE_2 t/\hbar}|2\rangle$

What if a particle is in a superposition of these two states?

Examples: – spin-½ in ~~the~~ magnetic field (consider in details next time)

– atomic levels and electron interacting with e-m radiation (will also consider later)

– neutrino's mass and flavor ($E = m_\nu c^2$, different masses, flavor: electron, muon, tau is the detection basis)

$|1\rangle$ —— $+\Delta E/2$

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$|2\rangle$ —— $-\Delta E/2$

If $|\alpha(t=0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$

↓ time evolution

$$|\alpha(t)\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle e^{i\Delta E t/2\hbar} + |2\rangle e^{-i\Delta E t/2\hbar}\right)$$

What is the probability of finding the system in the same state after time t?

$$P_+ = |\langle \alpha(t=0)|\alpha(t)\rangle|^2 = \frac{1}{4}\left|(\langle 1| + \langle 2|)\left(|1\rangle \cdot e^{i\Delta E t/2\hbar} + |2\rangle e^{-i\Delta E t/2\hbar}\right)\right.$$

$$= \frac{1}{4}\left| e^{i\Delta E t/2\hbar} + e^{-i\Delta E t/2\hbar}\right|^2 = \cos^2\left(\frac{\Delta E t}{2\hbar}\right) = \cos^2\left(\frac{(E_2 - E_1)t}{2\hbar}\right.$$

We can repeat the same steps to find the probability of finding the particle in the orthogonal state $\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$

$$P_- = \left| \frac{1}{\sqrt{2}} (\langle 1| - \langle 2|)\, |\alpha(t)\rangle \right|^2 =$$

$$= \frac{1}{4} \left| (\langle 1| - \langle 2|) \left(|1\rangle e^{i\Delta E t/2\hbar} + |2\rangle e^{-i\Delta E t/2\hbar} \right) \right|^2 =$$

$$= \frac{1}{4} \left| e^{i\Delta E t/2\hbar} - e^{-i\Delta E t/2\hbar} \right|^2 = \sin^2\left(\frac{\Delta E t}{2\hbar}\right)$$

Interestingly, the probabilities of finding the particle in a state $|1\rangle$ or $|2\rangle$ stay constant

$$P_{1,2} = |\langle 1,2|\, \alpha(t)\rangle|^2 = \frac{1}{2} \left| \underset{\text{or } \langle 2|}{\langle 1|} \left(|1\rangle e^{i\Delta E t/2\hbar} + |2\rangle e^{-i\Delta E t/2\hbar} \right) \right|^2$$

$$= \frac{1}{2} \left| e^{\pm i\Delta E t/2\hbar} \right|^2 = \frac{1}{2}$$

We may say that populations of the Hamiltonian eigenstates stays constant, but the coherence b/w them evolves in time.

Time evolution of the operator expectation values

Reminder: we can theoretically describe the quantum state evolution, but we can only measure the outcome of measurements $\longrightarrow$ operator expectation values

$$\langle \hat{A}(t)\rangle = \langle \alpha(t)|\hat{A}|\alpha(t)\rangle$$

In the hamiltonian eigenstates the expectation values of ~~the~~ an operator stay constant $\longrightarrow$ stationary states

$$|E(t)\rangle = e^{-iEt/\hbar}|E\rangle \; ; \quad \langle E(t)|e^{iEt/\hbar} = \langle E(t)|$$

$$\langle \hat{A}(t)\rangle \equiv \langle E(t)|\hat{A}|E(t)\rangle = e^{iEt/\hbar}\underbrace{\langle E|\hat{A}|E\rangle}_{\langle \hat{A}(t=0)\rangle} e^{-iEt/\hbar} = \langle \hat{A}(t=0)\rangle$$

However, this is not the case in general

$$\frac{d}{dt}\left[\langle \alpha(t)|\hat{A}|\alpha(t)\rangle\right] = \left[\frac{d}{dt}\langle \alpha(t)|\right]\hat{A}|\alpha(t)\rangle +$$

$$+ \langle \alpha(t)|\frac{d\hat{A}}{dt}|\alpha(t)\rangle + \langle \alpha(t)|\hat{A}\left[\frac{d}{dt}|\alpha(t)\rangle\right] =$$

Shrodinger eqn: $i\hbar\frac{d}{dt}|\alpha(t)\rangle = \hat{H}|\alpha(t)\rangle \Rightarrow \frac{d}{dt}|\alpha(t)\rangle = -\frac{i}{\hbar}\hat{H}|\alpha(t)\rangle$

$\frac{d}{dt}\langle \alpha(t)| = \frac{i}{\hbar}\langle \alpha(t)|\hat{H}^{+} = \frac{i}{\hbar}\langle \alpha(t)|\hat{H} \qquad \hat{H} = \hat{H}^{+}$

$$= \underbrace{\frac{i}{\hbar}\langle \alpha(t)|\hat{H}\hat{A}|\alpha(t)\rangle - \frac{i}{\hbar}\langle \alpha(t)|\hat{A}\hat{H}|\alpha(t)\rangle}_{\frac{i}{\hbar}\langle \alpha(t)|\hat{H}\hat{A}-\hat{A}\hat{H}|\alpha(t)\rangle \;=\; \frac{i}{\hbar}\langle \alpha(t)|[\hat{H},\hat{A}]|\alpha(t)\rangle} + \langle \alpha(t)|\frac{\partial\hat{A}}{dt}|\alpha(t)\rangle$$

To summarize:

$$\frac{d}{dt}\langle \hat{A}(t)\rangle = \frac{i}{\hbar}\langle \alpha(t)|\,[\hat{H},\hat{A}]\,|\alpha(t)\rangle + \langle \alpha(t)|\,\frac{\partial \hat{A}}{\partial t}\,|\alpha(t)\rangle$$

Thus even if the operator doesn't have explicite time dependence $\frac{d\hat{A}}{dt} = 0$

its expectation value will change in time if it does not commute with the hamiltonian