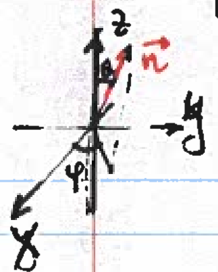

In classical physics $\vec{S} = \frac{\hbar}{2}\vec{n} = \frac{\hbar}{2}(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$$\langle S_z \rangle = \frac{\hbar}{2}\cos^2\theta/2 + \left(-\frac{\hbar}{2}\right)\sin^2\theta/2 = \frac{\hbar}{2}(\cos^2\theta/2 - \sin^2\theta/2)$$

$$= \frac{\hbar}{2}\cos\theta$$

$\langle S_x \rangle$ -?

$$|\vec{n}\rangle = \cos\theta/2\,|+z\rangle + e^{i\varphi}\sin\theta/2\,|-z\rangle =$$

$$= \cos\theta/2\left[\frac{1}{\sqrt{2}}|+x\rangle + \frac{1}{\sqrt{2}}|-x\rangle\right] + e^{i\varphi}\sin\theta/2\left[\frac{1}{\sqrt{2}}|+x\rangle - \frac{1}{\sqrt{2}}|-x\rangle\right] =$$

$$= \frac{1}{\sqrt{2}}\left[\cos\frac{\theta}{2} + e^{i\varphi}\sin\frac{\theta}{2}\right]|+x\rangle + \frac{1}{\sqrt{2}}\left[\cos\frac{\theta}{2} - e^{i\varphi}\sin\frac{\theta}{2}\right]|-x\rangle$$

Probability to measure $|+x\rangle$ if in the state $|\vec{n}\rangle$:

$$P_{|+x\rangle} = |\langle +x|\vec{n}\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} + e^{i\varphi}\sin\frac{\theta}{2}\right)\right|^2 =$$

$$= \frac{1}{2}\left(\cos\frac{\theta}{2} + e^{i\varphi}\sin\frac{\theta}{2}\right)\underbrace{\left(\cos\frac{\theta}{2} + e^{-i\varphi}\sin\frac{\theta}{2}\right)}_{\text{complex conjugate}}$$

$$= \frac{1}{2}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\underbrace{(e^{i\varphi} + e^{-i\varphi})}_{2\cos\varphi}\right) =$$

$$= \frac{1}{2}(1 + \sin\theta\cos\varphi)$$

$$P_{|-x\rangle} = |\langle -x|\vec{n}\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} - e^{i\varphi}\sin\frac{\theta}{2}\right)\right|^2$$

$$= \frac{1}{2}(1 - \sin\theta\cos\varphi)$$

$$\langle S_x \rangle = \frac{\hbar}{2}\cdot P_{|+x\rangle} + \left(-\frac{\hbar}{2}\right)P_{|-x\rangle} = \frac{\hbar}{2}\frac{1}{2}(1 + \sin\theta\cos\varphi) +$$

$$+ \left(-\frac{\hbar}{2}\right)\cdot\frac{1}{2}(1 - \sin\theta\cos\varphi) = \frac{\hbar}{2}\sin\theta\cos\varphi$$

$\langle S_y \rangle = \frac{\hbar}{2}\sin\theta\sin\varphi$, as expected from classical predictions

Important property of any quantum state: it~~s~~ can be multiplied to any phase factor $e^{i\delta}$ (where δ is any real number, so that $|e^{i\delta}|=1$), ~~and~~ without changing any measurable properties of this state

$$|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|+z\rangle \rightarrow |+y\rangle = -\frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle$$

or

$$|+y\rangle = \frac{i}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle$$

or

$$|+y\rangle = \frac{e^{i\delta}}{\sqrt{2}}|+z\rangle + \frac{ie^{i\delta}}{\sqrt{2}}|-z\rangle$$

$$P_{+z} = \left|\frac{1}{\sqrt{2}}\right|^2 = \left|-\frac{1}{\sqrt{2}}\right|^2 = \left|\frac{i}{\sqrt{2}}\right|^2 = \left|\frac{e^{i\delta}}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

same probability.

$$P_{-z} = \left|\frac{i}{\sqrt{2}}\right|^2 = \left|-\frac{i}{\sqrt{2}}\right|^2 = \left|-\frac{1}{\sqrt{2}}\right|^2 = \left|\frac{ie^{i\delta}}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Matrix

~~Vector~~ notation for quantum states

$$|\vec{n}\rangle = \cos\frac{\theta}{2}\,|+z\rangle + e^{i\varphi}\sin\frac{\theta}{2}\,|-z\rangle$$

$$\vec{n} = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}$$

explicite form (but long)

Instead, we can use vector components written as an element of a column

$$\vec{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \iff |\vec{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

we can do the same for quantum states

Because $|\pm z\rangle$ basis consists of only two vectors, any quantum state in this basis can be presented as a two-element column

$$|\alpha\rangle = c_+|+z\rangle + c_-|-z\rangle \qquad \langle\alpha| = c_+^*\langle+z| + c_-^*\langle-z|$$

or

$$|\alpha\rangle = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \qquad |+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

column

or

$$\langle\alpha| = \begin{pmatrix} c_+^* & c_-^* \end{pmatrix}$$

string

$$\langle+z| = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \langle-z| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Inner product

$$\langle\alpha|\alpha\rangle = \begin{pmatrix} c_+^* & c_-^* \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = c_+^*c_+ + c_-^*c_- = |c_+|^2 + |c_-|^2 = 1$$

$$|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle \qquad |-x\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle$$

or

$$|+x\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad |-x\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\langle+x| = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \qquad \langle-x| = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$|\pm y\rangle = \frac{1}{\sqrt{2}}|+z\rangle \pm \frac{i}{\sqrt{2}}|-z\rangle \quad \text{or} \quad |\pm y\rangle = \begin{pmatrix} 1/\sqrt{2} \\ \pm i/\sqrt{2} \end{pmatrix}$$

$$\langle\pm y| = \begin{pmatrix} 1/\sqrt{2} & \mp i/\sqrt{2} \end{pmatrix}$$

Example 1: what if we need to calculate inner product of $\langle +y|-x\rangle$?

"Old" method

$$\langle +y|-x\rangle = \left(\frac{1}{\sqrt{2}}\langle +z| - \frac{i}{\sqrt{2}}\langle -z|\right)\left(\frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle\right) =$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\underbrace{\langle +z|+z\rangle}_{=1} + \left(-\frac{i}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)\underbrace{\langle -z|-z\rangle}_{=1} = \frac{1}{2} + \frac{i}{2}$$

"New" method

$$\langle +y|-x\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{1}{2} + \frac{i}{2}$$

Example 2: more convenient algebra

$$|+z\rangle = \frac{1}{\sqrt{2}}|+x\rangle + \frac{1}{\sqrt{2}}|-x\rangle \Rightarrow \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|+z\rangle} = \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}_{|+x\rangle} + \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}}_{|-x\rangle}$$

In general if

$$|\alpha\rangle = \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \quad \text{and} \quad |\beta\rangle = \begin{pmatrix} b_+ \\ b_- \end{pmatrix}$$

$$|\gamma\rangle = c_\alpha|\alpha\rangle + c_\beta|\beta\rangle = c_\alpha\begin{pmatrix} a_+ \\ a_- \end{pmatrix} + c_\beta\begin{pmatrix} b_+ \\ b_- \end{pmatrix} =$$

$$= \begin{pmatrix} c_\alpha a_+ + c_\beta b_+ \\ c_\alpha a_- + c_\beta b_- \end{pmatrix}$$

$$c_\alpha = \langle\alpha|\gamma\rangle$$
$$c_\beta = \langle\beta|\gamma\rangle$$