Spin-spin interactions

Two possible bases:

Individual spins-1/2

$|1\rangle = |\uparrow\uparrow\rangle \quad |2\rangle = |\uparrow\downarrow\rangle$

$|3\rangle = |\downarrow\uparrow\rangle \quad |4\rangle = |\downarrow\downarrow\rangle$

operators $\longrightarrow$ 4×4 matrix

Total spin (eigenstates of S^2 and S_z)

$|s,m\rangle \quad s=0, \quad m=0$

$s=1, \quad m=0,\pm 1$

operators $\longrightarrow$ 4×4 matrix (different!)

$|1,1\rangle = |\uparrow\uparrow\rangle$, $|1,-1\rangle = |\downarrow\downarrow\rangle$

$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

Which basis to use? Either one will work to find the hamiltonian eigenstates, but sometimes one is more convenient

1. Non-interacting spins $\hat{H} = \omega_0 \hat{S}_{1z} + \omega_0 \hat{S}_{2z} = \omega_0 \hat{S}_z$

2. Interacting spins $\hat{H}_{int} = \frac{2A}{\hbar}\, \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$

Individual spins: $\hat{H}_{int} = \frac{2A}{\hbar^2}\left(\hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z}\right) =$

$= \frac{2A}{\hbar^2}\left(\frac{1}{2}\hat{S}_{1+}\hat{S}_{2-} + \frac{1}{2}\hat{S}_{1-}\hat{S}_{2+} + \hat{S}_{1z}\hat{S}_{2z}\right)$

Using these expressions we can fing matrix elements of $\hat{H}_{int}$ in $|1\rangle, |2\rangle, |3\rangle, |4\rangle$ basis

$\hat{H}|1\rangle = \frac{A}{2}|1\rangle \quad \hat{H}|4\rangle = \frac{A}{2}|4\rangle \quad \hat{H}|2\rangle = +A|3\rangle - \frac{A}{2}|2\rangle$

$\hat{H}|3\rangle = +A|2\rangle - \frac{A}{2}|3\rangle$

$$\hat{H} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} A/2 & 0 & 0 & 0 \\ 0 & -A/2 & A & 0 \\ 0 & A & -A/2 & 0 \\ 0 & 0 & 0 & A/2 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$H_{11} = \langle 1|\hat{H}|1\rangle = H_{44} = \langle 4|\hat{H}|4\rangle = \frac{A}{2}$

$H_{22} = H_{33} = -\frac{A}{2}$

$H_{23} = H_{32} = A$

Clearly, states $|1\rangle$ & $|4\rangle$ are the eigenstates of $\hat{H}$ $\quad \hat{H}|1\rangle = \frac{A}{2}|1\rangle \quad \hat{H}|4\rangle = \frac{A}{2}|4\rangle$

The states $|2\rangle$ & $|3\rangle$ are mixed!
Note that the central part of the Hamiltonian reminds us of the one we used for the flipping ammonia molecule!

$$\hat{H}_{\substack{int \\ |2\rangle,|3\rangle}} = \begin{pmatrix} -A/2 & A \\ A & -A/2 \end{pmatrix} \qquad \hat{H}_{amm} = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}$$

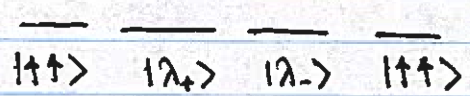
$\lambda_{\pm} = A/2, -3A/2$ $\qquad$ eigenvalues $\lambda_{\pm} = E_0 \pm A$

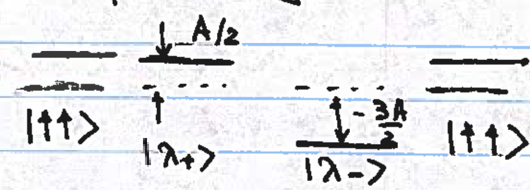
$|\lambda_+ = A/2\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)$ $\qquad |\lambda_{\pm}\rangle = \frac{1}{\sqrt{2}}(|1\rangle \mp |2\rangle)$

$|\lambda_- = -3A/2\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)$

Non-interacting spins
all states have same energy (degenerate)

Interacting spin
- degeneracy partially lifted





Alternative solution

$$\hat{\vec{S}} = (\hat{\vec{S}}_1 + \hat{\vec{S}}_2) \qquad \hat{S}^2 = (\hat{\vec{S}}_1 + \hat{\vec{S}}_2)^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\cdot\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2$$

$$\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2 = \frac{1}{2}\left[\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right]$$

Recall that $\hat{S}_1^2 = \frac{3\hbar^2}{4}\hat{\mathbb{1}}$ acting on any wavefunction of a spin 1/2 particle doesn't change it

$$\hat{S}_1^2\,|s,m\rangle = \frac{3\hbar^2}{4}\,|s,m\rangle \qquad \hat{S}_2^2\,|s,m\rangle = \frac{3\hbar^2}{4}\,|s,m\rangle$$

$$\hat{S}^2\,|s,m\rangle = \hbar^2 s(s+1)\,|s,m\rangle$$

$$\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2 = \frac{1}{2}\left[\hbar^2 s(s+1)\hat{\mathbb{1}} - \frac{3\hbar^2}{4}\hat{\mathbb{1}} - \frac{3\hbar^2}{4}\hat{\mathbb{1}}\right] = \frac{\hbar^2}{2}\left[s(s+1) - \frac{3}{2}\right]$$

$$\hat{H}\,|s=1,m\rangle = \frac{2A}{\hbar^2}\cdot\frac{\hbar^2}{2}\left[2 - \frac{3}{2}\right] = \frac{A}{2}$$

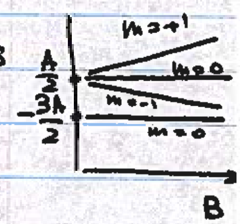
$$\hat{H}\,|s=0,m=0\rangle = \frac{2A}{\hbar^2}\cdot\frac{\hbar^2}{2}\left(-\frac{3}{2}\right) = -\frac{3A}{2}$$

3. Interacting spins in the magnetic field

$$\hat{H} = \omega_0\hat{S}_{1z} + \omega_0\hat{S}_{2z} + \frac{2A}{\hbar^2}\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2 = \omega_0\hat{S}_z + \frac{A}{\hbar^2}\left[\hat{S}^2 - \frac{3\hbar^2}{2}\hat{\mathbb{1}}\right]$$

still convenient to use $|s,m\rangle$ basis since these are still eigen states.

$\hat{H}\,|s=1,m\rangle = \hbar\omega_0 m + A/2$, $\hat{H}\,|s=0,m=0\rangle = -3A/2$



4. Positronium atom (+e) ~ (-e)

$$\hat{H} = \underbrace{\omega_0\hat{S}_{1z} - \omega_0\hat{S}_{2z}}_{\text{diagonal in } |1\rangle..|4\rangle \text{ basis}} + \underbrace{\frac{2A}{\hbar^2}\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2}_{\text{diagonal in } |s,m\rangle \text{ basis}} \leftarrow \text{neither is ideal}$$

Need to choose what basis, and honestly calculate matrix elements, and find eigenvalues.