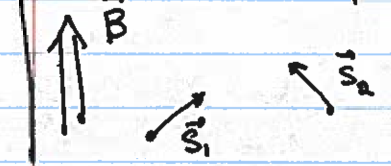
Two-particle systems

Two spins in an external magnetic field



$\hat{H}_0 = \omega_0 \hat{S}_{z_1} + \omega_0 \hat{S}_{z_2}$
non-interacting spins

However, each spin creates its own magnetic field, and the other spin senses it

$$\hat{H}_{int} = \frac{2A}{\hbar^2} \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 \qquad \text{spin-spin interaction.}$$

How to describe the system of two spins

Two options: – keep track of individul particle states
– measure their total spin

Two independent particle → basis $|\pm z\rangle_1$ & $|\pm z\rangle_2$

Spin 1: $|\alpha^{(1)}\rangle = c_1^{(1)} |S_1 = 1/2, m_1 = 1/2\rangle + c_2^{(1)} |S_1 = 1/2, m_1 = -1/2\rangle$
Spin 2: $|\alpha^{(2)}\rangle = c_1^{(2)} |S_2 = 1/2, m_2 = 1/2\rangle + c_2^{(2)} |S_2 = 1/2, m_2 = -1/2\rangle$

Two-particle state $|\alpha_{2p}\rangle = |\alpha^{(1)}\rangle \otimes |\alpha^{(2)}\rangle$
tensor product

Two single-particle wave functions are treated individually, not multiplied (just stuck together)

$|\uparrow\uparrow\rangle \equiv |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \qquad |+z,+z\rangle \equiv |+z\rangle_1 |+z\rangle_2$

If an operator is defined to act only on one particle, it acts only on the corresponding wavefunction, without affecting the other

$\hat{S}_{x_1} |+z\rangle_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-z\rangle_1$

$$\hat{S}_{x_1}|+z\rangle_1 \otimes |+z\rangle_2 = |-z\rangle_1 |+z\rangle_2$$

$$\hat{S}_{x_2}|+z\rangle_1 \otimes |+z\rangle_2 = |+z_1\rangle \otimes \left(\hat{S}_{x_2}|+z\rangle_2\right) = |+z\rangle_1 |-z\rangle_2$$

$$\hat{S}_{x_1}\hat{S}_{x_2}\, |+z\rangle_1 \otimes |+z\rangle_2 = \left(\hat{S}_{x_1}|+z\rangle_1\right) \otimes \left(\hat{S}_{x_2}|+z\rangle_2\right) =$$
$$= |-z\rangle_1 |-z\rangle_2$$

Total spin $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$

basis $|s, m\rangle$ – eisenstates of the operators $\hat{\vec{S}}^2$ and $\hat{S}_z$

In classical spin addition, the total spin can have any length from $|\vec{S}_1| + |\vec{S}_2|$ to $|\vec{S}_1 - \vec{S}_2|$. ~~Bas~~ If $|S_1| = |S_2| = \frac{\hbar}{2}$ any values from $2|S_1|$ $(= \hbar)$ to 0.

In Quantum spin addition only descreet values of total spin are possible

if $|S_1| = s_1$ and $|S_2| = s_2$ (or $|J_1| = j_1$ & $|J_2| = j_2$)

$s = s_1 + s_2,\ s_1 + s_2 - 1 \ldots |s_1 - s_2|$ or $j = j_1 + j_2, j_1 + j_2 - 1 \ldots |j_1 - j_2|$

if $s_1 = s_2 = 1/2$ $\quad s = 1$ or 0

If $j/s = 1$ $\quad m = 0, \pm 1$ $\quad$ three states (triplet)

if $j/s = 0$ $\quad m = 0$ $\quad$ single state (singlet)

Challenge: how to relate a quantum state of a two-spin system written in one-particle basis $(|\pm z\rangle_{1,2})$ to the total spin basis $|s, m\rangle$?

$$|s_1 = 1/2, m_1\rangle \otimes |s_2 = 1/2, m_2\rangle \equiv |m_1, m_2\rangle$$

$$\hat{J}_1^2 |m_1, m_2\rangle = \hat{J}_2^2 |m_1, m_2\rangle = \frac{3\hbar^2}{4} |m_1, m_2\rangle$$

$$\hat{J}_+|\uparrow\rangle = 0 \quad \hat{J}_+|\downarrow\rangle = \hbar|\uparrow\rangle \quad \hat{J}_-|\uparrow\rangle = |\downarrow\rangle \quad \hat{J}_-|\downarrow\rangle = 0$$

$$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_{+1}\hat{J}_{-2} + \hat{J}_{-1}\hat{J}_{+2} + 2\hat{J}_{z_1}\hat{J}_{z_2}$$

State $|\uparrow_1\uparrow_2\rangle \equiv |\uparrow\rangle \otimes |\uparrow\rangle$

$$\hat{J}^2|\uparrow_1\uparrow_2\rangle = \hat{J}_1^2|\uparrow\rangle|\uparrow\rangle + |\uparrow\rangle\hat{J}_2^2|\uparrow\rangle + \hat{J}_{+1}|\uparrow\rangle\hat{J}_{-2}|\uparrow\rangle$$

$$+ \hat{J}_{-1}|\uparrow\rangle\hat{J}_{+2}|\uparrow\rangle + 2\hat{J}_z|\uparrow\rangle\hat{J}_z|\uparrow\rangle = \frac{3\hbar^2}{4}|\uparrow\uparrow\rangle + \frac{3\hbar^2}{4}|\uparrow\uparrow\rangle +$$

$$+0 + 0 + 2\cdot\frac{\hbar}{2}\cdot\frac{\hbar}{2}|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle$$ eigenstate of $\hat{J}^2$

$$\hat{J}^2|j,m\rangle = j(j+1)|j,m\rangle \Rightarrow |\uparrow\uparrow\rangle \equiv |j=1, m=1\rangle$$

State $|\uparrow\downarrow\rangle \equiv |\uparrow\rangle \otimes |\downarrow\rangle$

$$\hat{J}^2|\uparrow\downarrow\rangle = \hat{J}^2|\uparrow\rangle|\downarrow\rangle + |\uparrow\rangle\hat{J}^2|\downarrow\rangle + \hat{J}_+|\uparrow\rangle\hat{J}_-|\downarrow\rangle + \hat{J}_-|\uparrow\rangle\hat{J}_+|\downarrow\rangle + 2\hat{J}_z|\uparrow\rangle J_z|\downarrow\rangle$$

$$= \frac{3\hbar^2}{4}|\uparrow\downarrow\rangle + \frac{3\hbar^2}{4}|\uparrow\downarrow\rangle + 0 + \underline{\underline{\hbar^2|\downarrow\uparrow\rangle}} - \frac{\hbar^2}{2}|\uparrow\downarrow\rangle$$

$$\hat{J}^2|\downarrow\uparrow\rangle = \frac{3\hbar^2}{4}|\downarrow\uparrow\rangle + \frac{3\hbar^2}{4}|\downarrow\uparrow\rangle + \underline{\underline{\hbar^2|\uparrow\downarrow\rangle}} + 0 - \frac{\hbar^2}{2}|\downarrow\uparrow\rangle$$

Individual states $|\uparrow\downarrow\rangle$ or $|\downarrow\uparrow\rangle$ are <u>not</u> eigenstates of $\hat{J}^2$, but their sum or difference are!

$$\hat{J}^2\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\right] = \underbrace{\left[\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + \frac{\hbar^2}{2} - \frac{\hbar^2}{2}\right]}_{2\hbar^2}\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

thus $\hat{J}^2\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\right] = 2\hbar^2[\dots] \Rightarrow j = 1$

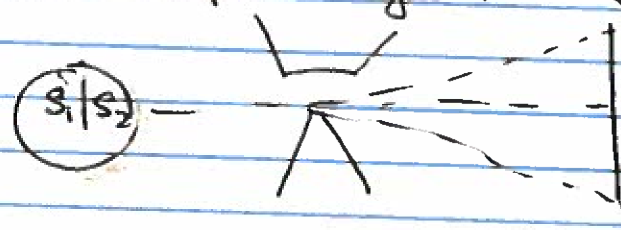
$$\hat{J}_z\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\right] = 0 \Rightarrow m = 0$$

so this is the $|j=1, m=0\rangle$ state!

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \longrightarrow \hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z} \longrightarrow m = m_1 + m_2$$

$$\hat{J}^2 = (\hat{J}_1 + \hat{J}_2)^2 = \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{J}_1 \cdot \hat{J}_2 = J_1^2 + J_2^2 + $$
$$+ 2\hat{J}_{1x}\hat{J}_{2x} + 2\hat{J}_{1y}\hat{J}_{2y} + 2\hat{J}_{1z}\hat{J}_{2z}$$

Remember our thought experiment about two spins "glued" together in a SG apparatus!



$+\hbar$ $|\uparrow\uparrow\rangle \rightarrow m=1, j=1$

0 $|\uparrow\downarrow\rangle$ or $|\downarrow\uparrow\rangle$ or combinations

$-\hbar$ $|\downarrow\downarrow\rangle \rightarrow m=-1, j=1$

$$|1,+1\rangle \equiv |+1/2\rangle_1 \otimes |+1/2\rangle$$

$$\hat{J}_z |1,+1\rangle = (\hat{J}_{z_1} + \hat{J}_{z_2}) |+1/2\rangle \otimes |+1/2\rangle = \left(\frac{\hbar}{2} + \frac{\hbar}{2}\right) |+\tfrac{1}{2}, +\tfrac{1}{2}\rangle$$

$$\hat{J}_z |1,+1\rangle = \hbar |1,+1\rangle$$

Similarly, $\hat{J}_z |1,-1\rangle = (\hat{J}_{z_1} + \hat{J}_{z_2}) |+1/2\rangle \otimes |+1/2\rangle = -\hbar |1,-1\rangle$

Tensor products of individual spins are eigen~~vects~~ states of $\hat{J}_z$, but <u>not</u> $\hat{J}^2$

We can simplify our lives a little if we use $\hat{J}_\pm$ operators

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \Rightarrow \hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-), \quad \hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)$$

$$\hat{J}_{x_1}\hat{J}_{x_2} = \frac{1}{4}(\hat{J}_{+1} + \hat{J}_{-1})(\hat{J}_{+2} + \hat{J}_{-2}) = \frac{1}{4}(\hat{J}_{+1}\hat{J}_{+2} + \hat{J}_{+1}\hat{J}_{-2} + \hat{J}_{-1}\hat{J}_{+2} + \hat{J}_{-1}\hat{J}_{-2})$$

$$\hat{J}_{y_1}\hat{J}_{y_2} = -\frac{1}{4}(\hat{J}_{+1} - \hat{J}_{-1})(\hat{J}_{+2} - \hat{J}_{-2}) = -\frac{1}{4}(\hat{J}_{+1}\hat{J}_{+2} + \hat{J}_{+1}\hat{J}_{-2} - \hat{J}_{-1}\hat{J}_{+2} + \hat{J}_{-1}\hat{J}_{-2})$$

$$\hat{J}_{x_1}\hat{J}_{x_2} + \hat{J}_{y_1}\hat{J}_{y_2} = \frac{1}{2}\hat{J}_{+1}\hat{J}_{-2} + \frac{1}{2}J_{-1}J_{+2}$$

$\hat{J}_+ |j,m\rangle = \hbar$ [illegible] $|j,m+1\rangle$ $\qquad$ $\hat{J}_- |j,m\rangle = \hbar$ [illegible] $|j,m-1\rangle$

What about $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ state?

$$\hat{J}^2 \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = \left[\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} - \hbar^2 - \frac{\hbar^2}{2}\right][\ldots] = 0$$

eigenstate $\Downarrow$ with $j=0$ and $m=0$

$$|0,0\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$