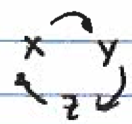
Spin -1/2 particles

Three spin components do not commute with each other

$$[\hat{S}_x, \hat{S}_y] = \hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x = i\hbar\hat{S}_z \, ; \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x \, ; \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$$



cyclic order

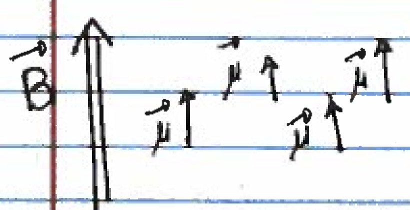
That means that ~~ha~~ none of the pairs can be measured with certainty at the same time

Uncertainty relations

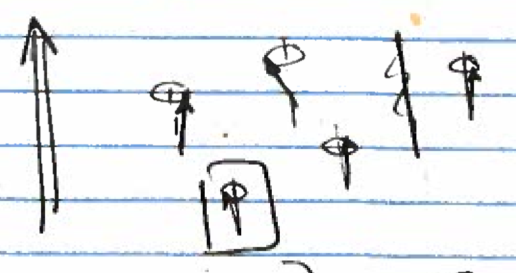
$$\Delta S_x \cdot \Delta S_y \geq \left(\frac{\hbar}{2}\right)^2$$ if a particle in $|+z\rangle$ state $\langle S_x\rangle = \langle S_y\rangle = 0$

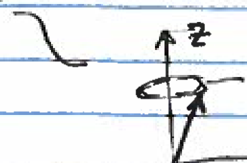
That means spins cannot be perfectly aligned

Classical view



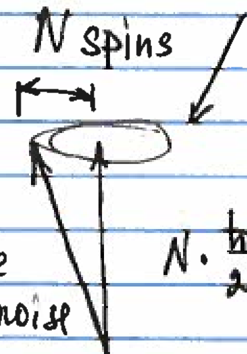
Quantum view





$\langle S_z\rangle = \frac{\hbar}{2}$

$\Delta S_x \approx \Delta S_y \approx \frac{\hbar}{2}$



uncertainty
$\sqrt{N}\cdot\hbar/2$

$N \cdot \frac{\hbar}{2} = S_z^{tot}$

spin projection noise
fundamental quantum noise

What about the length of the spin?

$$\hat{S}^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}\Big[\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}^2}_{=\hat{I}} + \underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}^2}_{=\hat{I}} + \underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}^2}_{=\hat{I}}\Big] = \frac{3\hbar^2}{4}\hat{I}$$

$\hat{S}^2$ commutes with each component
So, out of $\hat{S}^2, \hat{S}_x, \hat{S}_y, \hat{S}_z$ we can measure $\hat{S}^2$ and one of the components, usually $\hat{S}_z$

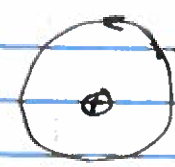
Common basis (i.e., the basis in which both operators have ~~can~~ same eigenvalues and eigenvectors) $|\pm z\rangle$

$$\hat{S}_z|\pm z\rangle = \pm\frac{\hbar}{2}|\pm z\rangle = \hbar\left(\pm\frac{1}{2}\right)|\pm z\rangle$$

$$\hat{S}^2|\pm z\rangle = \frac{3\hbar^2}{4}|\pm z\rangle = \hbar^2\,\frac{1}{2}\cdot\frac{3}{2}\,|\pm z\rangle = \hbar^2\,\frac{1}{2}\left(\frac{1}{2}+1\right)|\pm z\rangle$$

Brief interlude: in general,

$$\underbrace{\hat{\vec{J}}}_{\text{total angular momentum}} = \underbrace{\hat{\vec{S}}}_{\text{intrinsic angular momentum (spin)}} + \underbrace{\hat{\vec{L}}}_{\text{orbital angular momentum}}$$



$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$

You "met" $\hat{L}$ & $\hat{L}_z$ when discussing H-atom in PHYS201

$\underline{\underline{\psi_{n\ell m}}}$

$$\hat{L}^2\psi_{n\ell m} = \hbar^2\ell(\ell+1)\psi_{n\ell m}$$

$$\hat{L}_z\psi_{n\ell m} = \hbar m\,\psi_{n\ell m}$$

$$-\ell \le m \le +\ell$$

We are now going to introduce same notation for spins
Spin-1/2 particle will have $s = 1/2$, $m_s = \pm\frac{1}{2}$

Our familiar $|+z\rangle$ state becomes $|s=\frac{1}{2}, m_s=\frac{1}{2}\rangle$

$$|-z\rangle \longrightarrow |s=\tfrac{1}{2}, m_s=-\tfrac{1}{2}\rangle$$

$$\hat{S}^2|\pm z\rangle = \hat{S}^2|s=\tfrac{1}{2}, m_s=\pm\tfrac{1}{2}\rangle = \hbar^2 s(s+1)|s=\tfrac{1}{2}, m_s=\pm\tfrac{1}{2}\rangle = \frac{3}{4}\hbar^2|s=\tfrac{1}{2}, m_s=\pm\tfrac{1}{2}\rangle$$

$$\hat{S}_z|\pm z\rangle = \hat{S}_z|s=\tfrac{1}{2}, m=\pm\tfrac{1}{2}\rangle = \hbar m_s|s=\tfrac{1}{2}, m_s=\pm\tfrac{1}{2}\rangle = \pm\frac{\hbar}{2}|s=\tfrac{1}{2}, m_s=\pm\tfrac{1}{2}\rangle$$

Why do we need it? Because not all particles are spin 1/2!

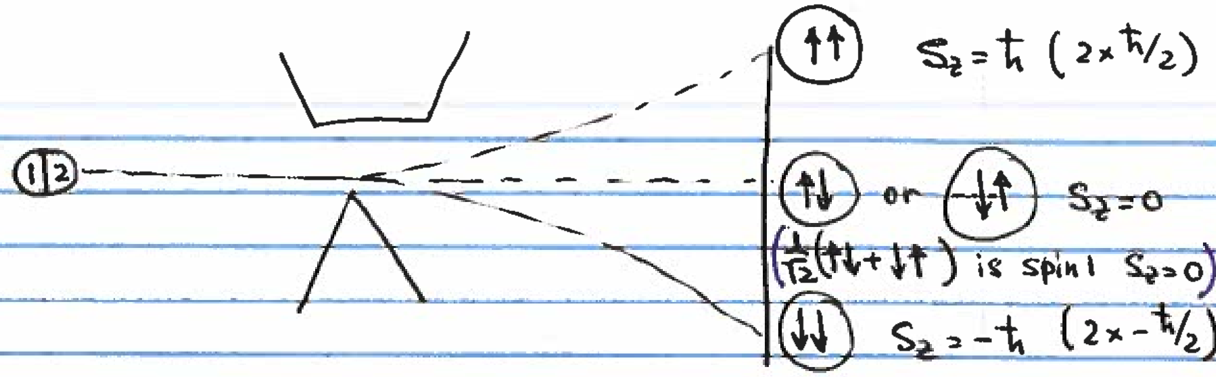
All basic elementary particles have spin-1/2 (electron, neutron, proton)

Photon is a spin-1 particle, but it is not affected by the magnetic field. Other particle that mediate interactions (like W-boson) often are spin-1 particles, but have short lifetime. So typically spin-1 particles we can experiment with are composites.

"Toy model" of a spin-1 particle

$\uparrow$ + $\uparrow$ trapped together
spin 1/2 spin 1/2

that is a good model, except it also includes a composite spin-zero particle



Switching back to $\hat{J}$ notation as per textbook

In $\hat{J}_z$ basis → three eigenvalues $(+\hbar, 0, -\hbar)$
Three eigenvectors

$$\hat{J}_z \begin{pmatrix}1\\0\\0\end{pmatrix} = \hbar \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad \hat{J}_z \begin{pmatrix}0\\1\\0\end{pmatrix} = 0\hbar \begin{pmatrix}0\\1\\0\end{pmatrix}; \quad \hat{J}_z \begin{pmatrix}0\\0\\1\end{pmatrix} = -\hbar \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\hat{J}_z = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & -1\end{pmatrix} \qquad \hat{J}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{pmatrix}$$

$$\hat{J}_y = \frac{\hbar}{2} \begin{pmatrix}0 & -i & 0\\i & 0 & -i\\0 & i & 0\end{pmatrix}$$

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = 2\hbar^2 \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \hbar^2 1\cdot(1+1)\,\hat{I}$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \equiv |j=1, m=1\rangle \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} = |j=1, m=0\rangle; \quad \begin{pmatrix}0\\0\\1\end{pmatrix} = |j=1, m=-1\rangle$$

$$\hat{J}^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle$$

$$\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle$$

For example, we can find expressions for $\hat{J}_y$ eigenstates. The full calculation (assuming unknown eigenvalues) is done in the textbook.

If we make a reasonabe assumption thas $\hat{J}_y$ has the same eigenvalues $\pm\hbar, 0$ as $\hat{J}_z$ (due to spatial symmetry)

Then

$$\hat{J}_y |y_+\rangle = \hbar |y_+\rangle, \quad \hat{J}_y |y_0\rangle = 0, \quad \hat{J}_y |y_-\rangle = \hbar |y_-\rangle$$

If $|y_+\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ $\quad \hat{J}_y |y_+\rangle = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \hbar \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$-i/\sqrt{2}\, c_2 = c_1$$
$$\frac{i}{\sqrt{2}}(c_1 - c_3) = c_2 \qquad |y_+\rangle = \begin{pmatrix} -i/\sqrt{2}\, c_2 \\ c_2 \\ i/\sqrt{2}\, c_2 \end{pmatrix}$$
$$i/\sqrt{2}\, c_2 = c_3$$

Normalization $\left|\frac{-i}{\sqrt{2}} c_2\right|^2 + |c_2|^2 + \left|\frac{ic_2}{\sqrt{2}}\right|^2 = 1$

$$\left(\frac{1}{2} + 1 + \frac{1}{2}\right)|c_2|^2 = 1$$

$$|c_2|^2 = 1/2$$

Often we choose the overall phase of a state such that the first coefficient is real and positive.

If we want that, we can pick $c_2 = i/\sqrt{2}$

$$|y_+\rangle = \begin{pmatrix} 1/2 \\ i/\sqrt{2} \\ -1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}$$

However, we can just as easily choose $c_2 = 1/\sqrt{2}$ and write

$$|\tilde{y}_+\rangle = \begin{pmatrix} -i/2 \\ 1/\sqrt{2} \\ i/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i \\ \sqrt{2} \\ i \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}$$

When we calculate any real values, both $|y_+\rangle$ and $|\tilde{y}_+\rangle$ give identical results

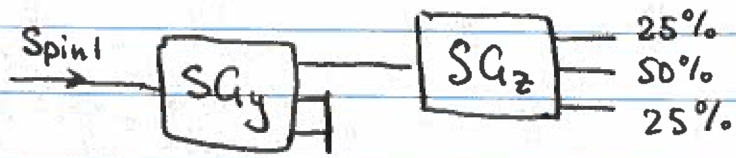
$$\hat{J}_y |y_0\rangle = 0 \qquad \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \qquad \begin{matrix} c_2 = 0 \\ c_1 = +c_3 \end{matrix}$$

$$|y_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ +1 \end{pmatrix}$$

for completeness

$$|y_-\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix}$$

Thus



same distribution for $|y_-\rangle$ state

but