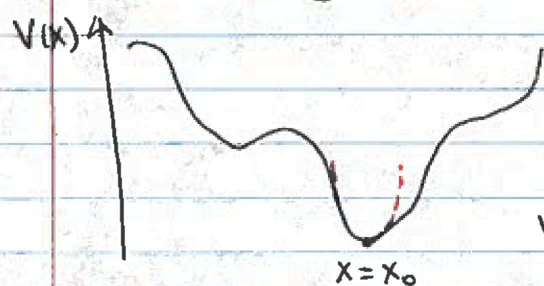
Simple Harmonic Oscillator

(almost as beloved by physisists as a spherical cow)

$$V(x) = \frac{1}{2} k x^2 = \frac{1}{2} m\omega^2 x^2$$

Why this potential is so important?
In most situations it describe the motion of a system near equilibrium

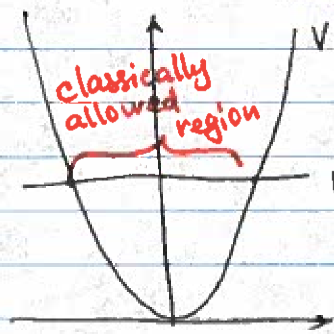


at the equilibrium

$$\frac{d}{dx} V(x=x_0) = 0$$

$$V(x) = V(x_0) + \cancel{\frac{dV}{dx}(x-x_0)}^{=0} + \underbrace{\frac{1}{2}\frac{d^2V}{dx^2}(x-x_0)^2}_{\text{leading term}} + \ldots$$

Spatial distribution of a particle in a harmonic potential



$$V(x) = \frac{1}{2} m\omega^2 x^2 \quad ; \quad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

Expectation: the wave function $\psi(x)$ oscillate within the classically allowed region, and decays outside

Shrodinger equation in x- basis

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 \psi = E\psi$$

Solving this equation will give us eigenstates and eigen energies of this Hamiltonian, its stationary states

Steps to solve this equation

① Move to the dimentional variable

$$x \to y = \sqrt{\frac{m\omega}{\hbar}} \cdot x \qquad \mathcal{E} = \frac{2E}{\hbar\omega}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi$$

transforms into

$$\frac{d^2\psi(y)}{dy^2} + (\mathcal{E} - y^2)\psi(y) = 0$$ only one free parameter left!

② Figure out asymptotic behavior

for $|y| \to \infty$ the solution must approach 0

if y is large $y^2 - \mathcal{E} \approx y^2$ (for finite $\mathcal{E}$)

@ $y \to \infty$ $$\frac{d^2\psi}{dy^2} - y^2\psi(y) = 0 \implies \psi(y) = A e^{-y^2/2}$$

($e^{y^2/2}$ is impossible)

③ Looking for a solution in a polynomial form (including the found asymptotic)

$$\psi(y) = h(y)\, e^{-y^2/2} \qquad \text{where} \qquad h(y) = \sum_{k=0}^{N} a_k y^k$$

$$\frac{d^2\psi(y)}{dy^2} + (\mathcal{E} - y^2)\psi(y) = 0$$

transforms into

$$\frac{d^2h}{dy^2} - 2y\frac{dh}{dy} + (\mathcal{E} - 1)h = 0$$

$$\downarrow$$

$$\sum_{k=0}^{N}\left[(k+2)(k+1)a_{k+2} - 2k a_k + (\mathcal{E} - 1)a_k\right] y^k = 0$$

(4) Obtain the recurrance relationship

$$\frac{a_{k+2}}{a_k} = \frac{2k+1-\varepsilon}{(k+2)(k+1)}$$

To keep the series finite, ε can have only very specific values $\varepsilon_n = 2n+1$ (then $\frac{a_{n+2}}{a_n} = 0$, so no ~~the~~ terms with $k>n$)

$$E_n = \frac{\hbar\omega}{2}\varepsilon_n = \hbar\omega(n+\tfrac{1}{2})$$

famous equidistant energy spectrum

$h_n(y)$ – Hermite polynomials

$H_0(y) = 1$ $\quad$ $H_2(y) = 4y^2-2$

$H_1(y) = 2y$ $\quad$ $H_3(y) = 8y^3-12y$

Eigen functions

$$\psi_n(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{2\hbar}}$$

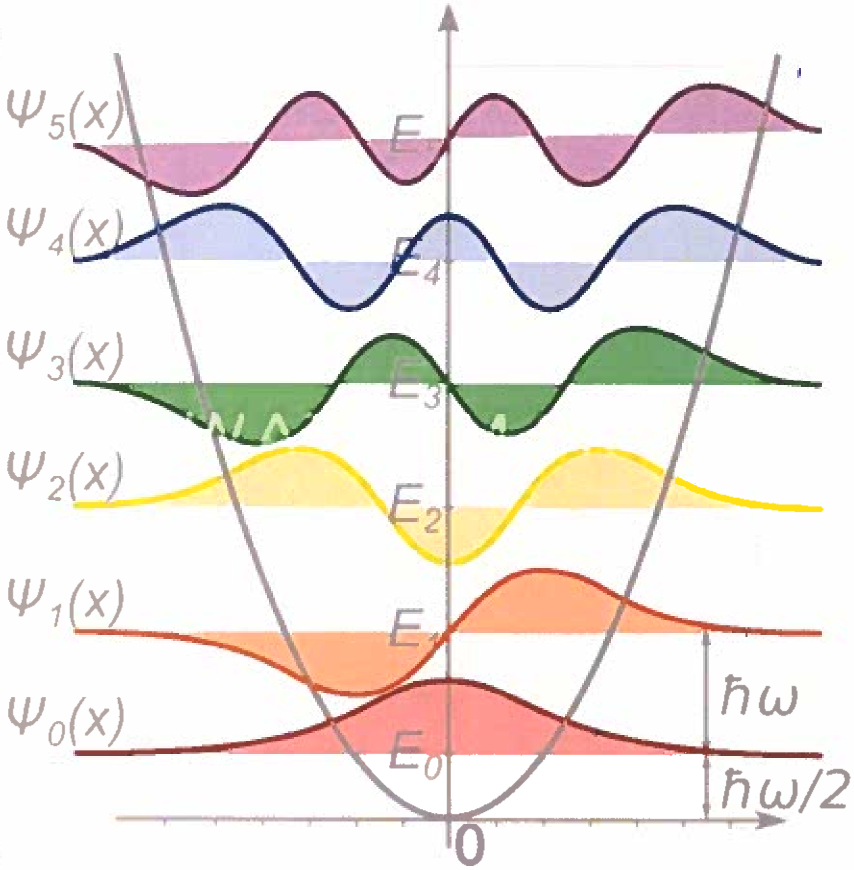
Ground state: $n=0$ $\quad$ $E_n = \frac{1}{2}\hbar\omega$

(zero-point energy)

$$\psi_0(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega x^2}{2\hbar}}$$

(gaussian distribution)

So to create a gaussian wave packet we need to trap a particle in a harmonic trap, and then let it go.



$\psi_5(x)$
$\psi_4(x)$
$\psi_3(x)$
$\psi_2(x)$
$\psi_1(x)$
$\psi_0(x)$
E_4
E_3
E_2
E_0
$\hbar\omega$
$\hbar\omega/2$
0

*Harmonic Oscillator Equation exists

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2}+\frac{1}{2}m\omega^2x^2\,\Psi(x)=E\Psi(x)$$

Mathematicians:

@exploring_interstellar

Very tough equation to solve, can't be solved by the algebraic method, Have yo use power series method and answer will be in Hermite polynomials... to long and tough

Meanwhile:

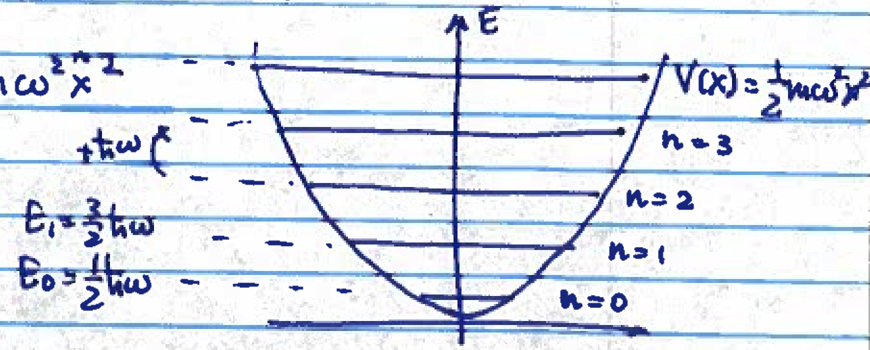
$$a=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)$$

$$a^{\dagger}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)$$

Simple harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

$E_n = \hbar\omega(n+\frac{1}{2})$
equidistant
spectrum



Ground state $n=0$ $E_0 = \frac{1}{2}\hbar\omega$ (zero-point energy)

$$\psi_0(x) = \langle x|0\rangle = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\, e^{-m\omega x^2/2\hbar}$$

In a classical world $E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 =$

$$= \frac{1}{2m}(p^2 + (m\omega x)^2) = \frac{1}{2m}(ip + m\omega x)(-ip + m\omega x)$$

We must be more careful when dealing with non-commuting operators $[\hat{x}, \hat{p}_x] = i\hbar$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right) \qquad \hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right)$$

(similar to $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$)

$$[\hat{a}, \hat{a}^+] = \frac{m\omega}{2\hbar}\left[(\hat{x} + \frac{i}{m\omega}\hat{p}_x)(\hat{x} - \frac{i}{m\omega}\hat{p}_x) - (\hat{x} - \frac{i}{m\omega}\hat{p}_x)(\hat{x} + \frac{i}{m\omega}\hat{p}_x)\right]$$

$$= \frac{m\omega}{2\hbar}\left[-\frac{i}{m\omega}\hat{x}\hat{p}_x + \frac{i}{m\omega}\hat{p}_x\hat{x} - \frac{i}{m\omega}\hat{x}\hat{p}_x + \frac{i}{m\omega}\hat{p}_x\hat{x}\right] =$$

$$= -\frac{i}{\hbar}[\hat{x}\hat{p}_x - \hat{p}_x x] = -\frac{i}{\hbar}[\hat{x}, \hat{p}] = -\frac{i}{\hbar}\cdot i\hbar = 1$$

$$[\hat{a}, \hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$$

$$\hat{a}\hat{a}^+ = \hat{a}^+\hat{a} + 1$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^+) \qquad \hat{p}_x = -i\sqrt{\frac{m\omega\hbar}{2}}\,(\hat{a} - \hat{a}^+)$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = -\frac{\hbar\omega}{4}(\hat{a} - \hat{a}^+)^2 + \frac{\hbar\omega}{4}(\hat{a} + \hat{a}^+)^2 =$$

$$= -\frac{\hbar\omega}{4}(\hat{a}^2 - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} + \hat{a}^{+2}) + \frac{\hbar\omega}{4}(\hat{a}^2 + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}^{+2})$$

$$= \frac{\hbar\omega}{2}(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}) = \frac{\hbar\omega}{2}(2\hat{a}^+\hat{a} + 1) = \hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2})$$

$$\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2}) \qquad E_n = \hbar\omega(n + \tfrac{1}{2})$$

Number operator $\quad \hat{n} = \hat{a}^+\hat{a} \qquad \hat{H} = \hbar\omega(\hat{n} + \frac{1}{2})$

Clear physial meaning: the energy of the system is its zero-point energy + $n \times$ added energy quanta

Eigenstates corresponding to E_n: $|n\rangle$, $\hat{n}|n\rangle = n|n\rangle$

$$\langle n|\hat{H}|n\rangle = E_n \Rightarrow \langle n|\hbar\omega(\hat{n} + \tfrac{1}{2})|n\rangle = \hbar\omega(n + \tfrac{1}{2})$$

What exactly $\hat{a}$ and $\hat{a}^+$ do?

$$\hat{a}|n\rangle = |?\rangle$$

$$\hat{H}|?\rangle = \hat{H}\hat{a}|n\rangle = \hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2})\hat{a}|n\rangle =$$

$$= \tfrac{1}{2}\hbar\omega\,\hat{a}|n\rangle + \hbar\omega\,\underbrace{\hat{a}^+\hat{a}\hat{a}}_{(\hat{a}\hat{a}^+ - 1)\hat{a}}|n\rangle = \tfrac{1}{2}\hbar\omega\,\hat{a}|n\rangle - \hbar\omega\,\hat{a}|n\rangle + \hbar\omega\,\hat{a}\,\underbrace{\hat{a}^+\hat{a}|n\rangle}_{\hat{n}|n\rangle = n|n\rangle} =$$

$$= -\tfrac{1}{2}\hbar\omega\,\hat{a}|n\rangle + n\hbar\omega\,\hat{a}|n\rangle = \hbar\omega\left[(n-1) + \tfrac{1}{2}\right]\underbrace{\hat{a}|n\rangle}_{|?\rangle}$$

If $\hat{H}|?\rangle = \hbar\omega\left[(n-1)+\frac{1}{2}\right]|?\rangle$

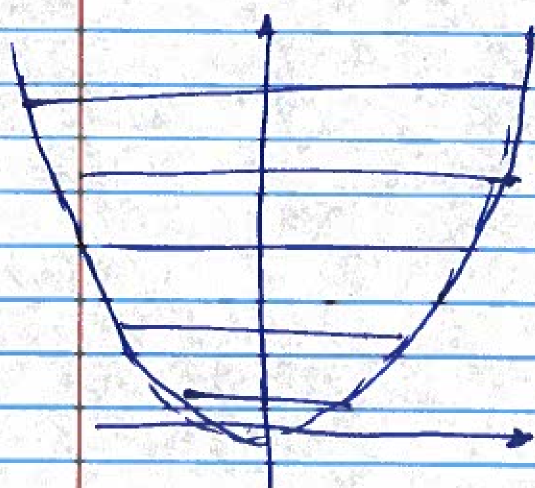
Then $|?\rangle$ is same as $|n-1\rangle$, with some coefficient

In fact

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \text{lowering operator}$$

similarly

$$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \text{raising operator}$$



$|n\rangle$

$$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{a}^{-}|n\rangle = \sqrt{n}\,|n-1\rangle$$

sometimes called ladder operators

We can also use this formalism to figure out $\psi_n(x)$

$$\hat{a}|0\rangle = 0 \qquad \langle x|\hat{a}|0\rangle = \langle x|\hat{x} + \frac{i}{m\omega}\hat{p}_x|0\rangle = 0$$

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_0(x) = 0 \qquad \frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar}x\psi_0$$

$$\psi_0(x) = A e^{-\frac{m\omega x^2}{2\hbar}} \xrightarrow{\text{normalize}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$|1\rangle = \hat{a}^{+}|0\rangle$$

$$|n+1\rangle = \frac{1}{\sqrt{n+1}}\hat{a}^{+}|n\rangle = \frac{1}{\sqrt{(n+1)n}}(\hat{a}^{+})^2|n-1\rangle =$$

$$= \frac{1}{\sqrt{(n+1)!}}(\hat{a}^{+})^{n+1}|0\rangle$$

$$|\psi_n\rangle = \cancel{\frac{1}{\sqrt{n!}}} \qquad |n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n\,|0\rangle$$

$$\psi_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{n!}}\left(\sqrt{\frac{m\omega}{2\hbar}}\right)^n \left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n \psi_0(x)$$